Incompatibility,Non-locality, and Tensor Norms

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1. Introduction

Quantum mechanics has several non-classical features which have received a lot of attention throughout its history, mainly due to the insight they bring and to their potential applications in quantum communication, cryptography, and computing. Among those, incompatibility of quantum measurements and Bell non-locality stand out. These two notions are directly related by the following simple observation: in a Bell non-locality experiment, if the measurements performed are compatible, one cannot observe a violation of Bell inequalities. Due to the paramount importance of Bell inequalities, there has been a lot of interest recently in the converse implication: does incompatibility imply a violation of a given Bell inequality? The first result in this direction is an equivalence of the two notions for the CHSH inequality [1]. Further work showed that this is no longer the case in more general scenarios. The key insight of our work can be stated as follows: The relation between *quantum measurement incompatibility* and *violations of Bell inequalities* can be naturally stated in the framework of the metric theory of tensor product of Banach spaces.

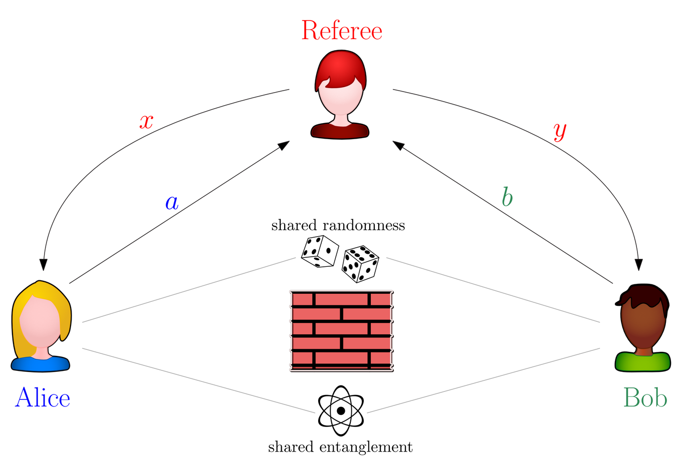
2. Compatibility [2]

Two POVMs A_i , B_j on M_d are *compatible* iff their exist a third POVM $C_{i,j}$ such that:

$$\forall i \in [k], \qquad A_i = \sum_{j=1}^{l} C_{ij}$$

$$\forall j \in [l], \qquad B_j = \sum_{i=1}^{k} C_{ij}.$$

3. Non-locality[3]



- *The classical value of the game M*:

$$\omega(M) := \sup_{\gamma_{x,y} \in \mathbb{L}} \left| \sum_{x,y=1}^{N} M_{x,y} \gamma_{x,y} \right|$$

with $\gamma_{x,y} = \int_\Lambda a_x(\lambda) b_y(\lambda)\, \mathrm{d}\mu(\lambda)$

- *The quantum value of the game M*:

$$\omega^*(M) := \sup_{\gamma_{x,y} \in \mathbb{Q}} \left| \sum_{x,y=1}^{N} M_{x,y} \gamma_{x,y} \right|$$

with $\gamma_{x,y} = \left\langle \psi \middle| A_x \otimes B_y \middle| \psi \right\rangle$

4.Compatibility and Non locality[4]

Fix any N-input, 2-output *invertible non-local game* $M \in \mathcal{M}_N(\mathbb{R})$, and consider the scenario where Alice's N dichotomic measurements $(A_1, A_2, \ldots, A_N)$ are fixed.
Definition (*Compatibility (tensor) norm*):
Alice construct $A = \sum_{i=1}^{N} |i\rangle \otimes A_i$ within her measurement operators.
The measurements $(A_1, A_2, \ldots, A_N)$ are compatible $\iff$ $\|A\|_c \leq 1$, with

$$\|A\|_c := \inf \left\{ \left\| \sum_{j=1}^{K} H_j \right\|_\infty \, : \, A = \sum_{j=1}^{K} z_j \otimes H_j, \;\; \text{s.t.} \;\; \|z_j\|_\infty \leq 1 \;\text{ and }\; H_j \geq 0 \right\}$$

where $z_j \in \mathbb{R}^N$ and $H_j \in \mathcal{M}_d^{sa}(\mathbb{C})$; $\|\cdot\|_c$ is called the *compatibility norm*.
Definition (*Non-locality (tensor) norm*):
For a N-tuple of measurement operators on Alice's side $(A_1, \ldots, A_N)$, the largest quantum value of the game M defines a tensor norm

$$\|A\|_M := \sup_{\|\psi\|=1} \sup_{\|B_y\| \leq 1} \left\langle \psi \middle| \sum_{x,y=1}^{N} M_{xy} A_x \otimes B_y \middle| \psi \right\rangle = \lambda_{\max} \left[\sum_{y=1}^{N} \left| \sum_{x=1}^{N} M_{x,y} A_x \right| \right]$$

Alice's measurements are called *M-Bell-local* if for any choice of Bob's observables B and for any shared state ψ, one cannot violate the Bell inequality corresponding to M:

$$\|A\|_M \leq \omega(M)$$

Otherwise, they are called *M-Bell-non-local.*

5. Main theorems[4]

Theorem 1:
Compatible measurements on Alice's side cannot yield a violation of the Bell inequality corresponding to M. Quantitatively, for all N-tuples of dichotomic measurements $A = (A_1, \ldots, A_N)$, we have

$$\|A\|_M \leq \|A\|_c \cdot \omega(M)$$

This first theorem describes that if Alice observe a violation of Bell inequality then her measurements are incompatible.

The *reverse inequality* holds, up to a constant depending on the game M.
Theorem 2:
For all invertible Bell inequalities $M \in \mathcal{M}_N(\mathbb{R})$ and all A, it holds that

$$\|A\|_c \leq \|A\|_M \cdot \max \left\{ |(M^{-1})_{x,y}| \right\}_{x,y=1}^{N}$$

Theorem 3:
For all invertible non-local games $M \in \mathcal{M}_N(\mathbb{R})$, we have

$$\omega(M) \cdot \max \left\{ |(M^{-1})_{x,y}| \right\}_{x,y=1}^{N} \geq 1$$

with equality if and only if $N = 2$ and M is a permutation of $M_{\text{CHSH}} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

6. Conclusions

In our work we introduced the *compatibility norm* and *Non-locality norm.* We give the quantitative description of the link between the compatibility of quantum measurements for dichotomic POVMs and non locality in the same framework of tensor norms for general invertible games,and M_{CHSH} Bell inequality is essentially the only one which characterizes measurement incompatibility in the scenario where Alice's measurements are fixed

7. References

[1] Michael M Wolf, David Perez-Garcia, and Carlos Fernandez. Measurements incompatible in quantum theory cannot be measured jointly in any other no-signaling theory. *Physical review letters*, 103(23):230402, 2009. 2

[2] Teiko Heinosaari, Takayuki Miyadera, and Mário Ziman. An invitation to quantum incompatibility. *Journal of Physics A: Mathematical and Theoretical*, 49(12):123001, Feb 2016. 2

[3] Carlos Palazuelos and Thomas Vidick. Survey on nonlocal games and operator space theory. *Journal of Mathematical Physics*, 57(1):015220, Jan 2016. 2

[4] Faedi Loulidi and Ion Nechita. Measurement incompatibility vs. bell non-locality: an approach via tensor norms, 2022. 2