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List of contributed articles

This thesis is based on the following published articles:

Loulidi, Faedi, and Ion Nechita. ”The compatibility dimension of quantum measurements.” Journal of Mathematical Physics 62.4 (2021): 042205.
(See also article [1] in the bibliography)

Loulidi, Faedi, and Ion Nechita. ”Measurement Incompatibility versus Bell Nonlocality: An Approach via Tensor Norms.” PRX Quantum 3.4 (2022): 040325.
(See also article [2] in the bibliography)

Contents

Chapter 1

Introduction

What is the nature of the reality of the world around us? What is space, time and matter? What are their origins? To find an answer to these questions is the deep desire of man to understand his existence and environment. Physics provides some answers to these conceptual questions. The discovery of the subatomic world reveals the unsuspected existence of a physical reality that is both rich and conceptually counterintuitive and mysterious.

Through quantum physics, our classical perception of reality is completely turned upside down. Now in the information era, quantum information will have a conceptual repercussion on the foundations of physics. The purpose of this manuscript is to understand two fundamental concepts of quantum physics through the lens of information, the compatibility of quantum measurements and nonlocality.

The discovery of quantum theory in the last century led to significant development in understanding physical reality. One of the first results of this theory is to understand the stability and the existence of matter. Historically two different approaches to the quantum theory are established which are today known as the Schrödinger and the Heisenberg formulations. In the Schrödinger approach [3], we can associate to a physical quantum system a *wave function* which is a solution of the renowned Schrödinger equation. With this solution,we can deduce the configuration probability of the system. The Heisenberg formulation of the quantum theory is described by matrices or operators and is known as *matrix mechanics* [4]. The two approaches lead to the exact physical predictions and the unification of the two formulations in the modern and complete mathematical framework based on Hilbert space was established by von Neumann [5, 6] and Dirac [7, 8].

In this thesis, we shall focus on two fundamental concepts of the quantum theory, the (in)compatibility of quantum measurements, Bell inequality violations, and the connections between the two. It is well known that one of the fundamental differences between quantum theory and classical physics is the existence of incompatible measurements. We say that two measurements are compatible if we can measure them at the same time; if they are not, we say they are incompatible. The other notion we shall address in this thesis is the nonlocality of quantum theory: at the quantum level the locality principle is not respected. In the first part, of this thesis from Chapter 2 to Chapter 7, we give the essential material needed for the second part. Chapter 8 and 9, are based on the articles produced during this thesis. In Chapter 2, we will give a brief introduction to quantum information theory, we will recall the axioms of the quantum theory, the density matrix formalism generalizing the pure state description, the fundamental concept of quantum channels, and we will end with two different approaches of the quantum theory the algebraic quantum theory and generalized probabilistic theories (GPT). In Chapter 3, we will give a brief introduction to the theory of tensor norms on Banach spaces. We will start by recalling the abstract definition of Banach spaces, then we will introduce the theory of tensor products of Banach spaces. This chapter will play a crucial role in the nonlocality framework and in understanding the link between incompatibility and nonlocality. In Chapter

4, we give an introduction to nonlocality and Bell inequalities, we introduce the framework of nonlocal games, and we end this chapter by showing the intrinsic link between nonlocality and tensor norms.

Chapter 5 is based on [1], where we shall introduce the standard notion of compatibility. We will introduce the new notion of compatibility dimension, which is used to analyze the effect of the Hilbert space dimension on the compatibility of quantum measurements. Intuitively, the compatibility dimension asks if for a given tuple of incompatible measurements in a given Hilbert space, can we find a subspace with reduced dimension so that they become compatible. This question was analyzed in [1] where several bounds were obtained with illustrative examples. We will introduce different types of noise models known in the literature that make incompatible measurements compatible by constructing noisy measurements given as a convex combination of the original measurement and a trivial operator. We will explore also the link between the noise model and approximate quantum cloning. As an application of the compatibility dimension, we will see that certain incompatible noisy projective measurements constructed from mutually unbiased bases become compatible if one reduces the Hilbert space dimension. More precisely, one can find a subspace of dimension less than the square root of the dimension of the original Hilbert space and an isometry constructed from a third mutually unbiased basis that makes the noisy projective mutually unbiased basis measurement compatible. We conclude this chapter by introducing the formulation of compatibility with the tensor norm formalism known as the compatibility (tensor) norm. The compatibility norm was first introduced in the context of generalized probabilistic theories, where a formulation of compatibility was introduced and analyzed with the tensor norm framework. If one reduces the generalized probabilistic theory formalism to the usual quantum mechanical one, we obtain a compatibility tensor norm framework in the quantum mechanical setting. The compatibility norm will become relevant in analyzing the intrinsic link between incompatibility and nonlocality.

Chapter 6 is based on [2], where we made the link between (in)compatibility of quantum measurements and nonlocality. It was shown in [9] that incompatibility is equivalent to Bell inequality violation for the CHSH game, and the question remained open if this equivalence holds for other games. In this chapter, we will go beyond the CHSH game, for that we will take the point of view of nonlocal games. In this setting, to analyze the effect of the incompatibility of Alice's measurement on the nonlocal effects, we fix Alice's measurements. From her measurements, she will construct a tensor and compute the G-Bell-(non)locality norm and the compatibility tensor norm. The new notion of the G-Bell-(non)locality captures the violation of a Bell inequality corresponding to the game G. The G-Bell-(non)locality norm is computed by optimizing Bob's measurements over the shared quantum state. We say that Alice's measurements are G-Bell-local if it is less than the classical value of the game, which is the maximal expectation of winning the game in the classical setting. We have shown, using some inequalities, that the compatibility tensor norm and the G-Bell-(non)locality norm are, in general, not equal. However, in [9], for the CHSH game, the authors have shown that they are equivalent; by translating this result in our context, we see that the two norms are equal. With the strong equivalence in the sense of [9], we have shown that with sufficient conditions, the only game satisfying this equivalence is the CHSH game. In Chapter 7, we will conclude the thesis by reviewing the contributions, and we will end with some open questions and future research directions.

Part I

Introduction and results

Chapter 2

Quantum information theory

The contradiction between experiments and the well-established theories at the beginning of the twentieth century, like the black body radiation, led Max Planck to the discovery of the quantum theory. Several developments and applications of quantum theory were explored in the last century. This has led to the discovery of the transistor, which is an important technological component in all our devices used today. Despite the all astonishing applications of quantum theory today, many fundamental concepts and questions are not yet answered. Nowadays quantum theory proceeds to a second quantum revolution in the era of information where understanding how information propagates inside a quantum physical device becomes very crucial. In this sense, for a given physical quantum system, what are the possible operations or protocols to observe certain phenomena? Knowing what are the different possibilities that can be obtained from the information encoded in a physical system is at the heart of the current development of quantum information theory.

In this chapter, we shall give a brief overview of concepts and tools used in quantum information theory. For that, in Section 2.1 we shall recall the postulates of the quantum theory in the setting of pure states. In Section 2.2 we shall recall the postulates of the quantum theory in the density matrix formalism. In Section 2.3 we will introduce the composite system framework. In Section 2.4 we will introduce the formalism of quantum channels that plays a crucial role in quantum information theory. In Section 2.5 we will introduce different approaches to quantum theory where we will restrict ourselves to the algebraic approach, and the GPT approach. This material can be found in standard textbook [10–18] and the lectures [19, 20].

2.1 Quantum postulates

At the microscopic level, nature is governed by quantum mechanical principles. In the following section, we shall recall the basic standard principles of quantum mechanics in finite dimension[1].

2.1.1 Quantum state

To a physical system $\mathcal{S}$, we associate a complex finite-dimensional Hilbert space $\mathcal{H}$. Since for a system with a finite number of levels, the Hilbert space of such systems is of finite dimension, we shall assume that $\mathcal{H} \cong \mathbb{C}^d$. But for more complicated physical systems like the harmonic oscillator, light, and spin systems, the Hilbert spaces are infinite-dimensional.

A quantum state is a normalized vector $|\psi\rangle$ on $\mathcal{H}$ that encodes all the *information* of the physical system. Two vectors $|\psi\rangle$ and $|\varphi\rangle$ on $\mathcal{H}$ describe the physical system if

$$|\psi\rangle = \lambda \, |\varphi\rangle$$

[1] All the postulates hold in infinite dimensional spaces describing continuous systems.

with $\lambda \in \mathbb{C}^*$, and by the normalisation condition the vectors $|\psi\rangle$ and $|\varphi\rangle$ are defined up to *a global phase*.

2.1.2 Observable

Observables in quantum mechanics that characterize physical measurable quantity like the energy of the system are described by selfadjoint operators $A = A^* \in \mathcal{B}(\mathcal{H}) \cong \mathcal{M}_d(\mathbb{C})$.

In finite-dimensional Hilbert spaces, the spectral decomposition of an observable A is given by

$$A = \sum_i \lambda_i P_i,$$

where $\lambda_i \in \mathbb{R}$ are the eigenvalues of A and P_i are orthogonal projections[2]*. The probability distribution of obtaining the outcome λ_i when measuring the observable A on a quantum system in a state $|\psi\rangle$ is given by the* Born rule*:*

$$\mathbb{P}(\lambda_i) := \| P_i |\psi\rangle \|^2,$$

The measurement of the outcome λ_i induces a change of the quantum state given by

$$|\psi'\rangle := \frac{P_i |\psi\rangle}{\| P_i |\psi\rangle \|}.$$

This is known as the wave function collapse.

2.1.3 Dynamics

The evolution of the quantum systems is governed by a unitary matrix $U \in \mathcal{U}(\mathcal{H}) : |\psi'\rangle = U|\psi\rangle$. An initial state $|\psi(t_0)\rangle$ is evolved at a time t to a state $|\psi(t)\rangle$ where we have

$$|\psi(t)\rangle := U(t, t_0) |\psi(t_0)\rangle .$$

The unitary matrix $U(t, t_0)$, is given by

$$U(t, t_0) = e^{-iH(t-t_0)},$$

where $H \in \mathcal{B}(\mathcal{H})$ is a self-adjoint operator known as the Hamiltonian of the quantum system. This is known as the Schrödinger picture; equivalently there exists another representation where the evolution is on the observables and the quantum state is fixed (this is known as the Heisenberg picture).

2.2 Density matrix formalism

Generally, a quantum system is completely described by *density matrices* that we shall denote by ρ. The density matrix formalism was established historically to describe a statistical mixture with a probability distribution of pure states. Density matrices are *positive semi-definite matrices of trace one*. We shall denote the set of such matrices by $\mathcal{M}_d^{1,+}$[3] which is given by

$$\mathcal{M}_d^{1,+} := \Big\{ \rho \in \mathcal{M}_d(\mathbb{C}) \, : \, \rho \geq 0 \, ; \, \operatorname{Tr} \rho = 1 \Big\}.$$

The mathematical formalism with density matrices or the mixed state description generalizes the one introduced previously with the ket vectors or the pure state formalism. The set of density

[2]Also the projectors are known in the literature as von Neumann measurement or projective valued measure or simply PVM.

[3]Generally, they are *trace-class operators* $\mathcal{T}(\mathcal{H})$; see Subsection 2.5.1.

matrices is a *convex set*, where its extreme points are just projectors of rank one [10, Proposition 2.11]. We shall denote the projectors as $|\psi\rangle\langle\psi|$, where $\langle\psi|$ is the dual vector of $|\psi\rangle$.

As before, the evolution of the quantum system is given by a unitary $U \in \mathcal{U}(\mathcal{H})$, where the evolution of a state ρ_0 at t_0 to ρ_t at t is given by

$$\rho_t = U(t, t_0)\,\rho_0\,U^*(t, t_0).$$

The probability of observing the outcome λ_i, the eigenvalue of an observable A is given by

$$\mathbb{P}(\lambda_i) = \mathrm{Tr}(P_i\,\rho) = \langle P_i\,, \rho\rangle_{HS}.$$

where we have used the *Hilbert-Schmidt* scalar product defined on $\mathcal{M}_d(\mathbb{C})$ as follows:

$$\begin{aligned}\langle\cdot,\cdot\rangle_{HS} : \mathcal{M}_d(\mathbb{C}) \times \mathcal{M}_d(\mathbb{C}) &\to \mathbb{C}, \\ (A, B) &\to \langle A\,, B\rangle_{HS} := \mathrm{Tr}(A^*\,B).\end{aligned}$$

The measurement procedure induces a change in the quantum state, where the resulting quantum state is given by

$$\rho' = \frac{P_i\,\rho\,P_i}{\mathrm{Tr}(P_i\,\rho)}.$$

Actually, those types of measurements are very specific and are known as projective measurement, but in general, the measurement process is described by *Positive operator Valued Measure* or briefly *POVM*. The POVMs elements are positive operators that sum up to the identity and we have the following definition

Definition 2.2.1. *A* positive operator valued measure *(POVM) on $\mathcal{M}_d$ with k outcomes is a k-tuple $A = (A_1, \ldots, A_k)$ of self-adjoint operators from $\mathcal{M}_d$ which are positive semidefinite and sum up to the identity:*

$$\forall i \in [k]^4, \quad A_i \geq 0 \qquad \textit{and} \qquad \sum_{i=1}^{k} A_i = I_d.$$

When measuring a quantum state ρ with the apparatus described by A, we obtain a random outcome from the set $[k]$:

$$\forall i \in [k], \qquad \mathbb{P}(\textit{outcome} = i) = \mathrm{Tr}[\rho A_i].$$

The vector of outcome probabilities $(\mathrm{Tr}[\rho A_i])_{i=1}^k$ is indeed a probability vector; note that the properties of the operators A_i, called *quantum effects*, are tailor-made for this. This mathematical formalism used to describe quantum measurements (or POVMs, or *meters*) does not account for what happens with the quantum particle after the measurement. One can think that the particle is destroyed in the process of measurement (see Figure 2.1) and thus only the outcome probabilities are relevant. In this thesis, we shall use the POVM formalism since it is the most general way of extracting classical information from a quantum system.

2.3 Composite systems

In this section, we shall introduce the tensor product structure of quantum mechanics. Such structure arises when one wants to describe a composite system of two subsystems or more. More precisely, we consider a physical system $\mathcal{S}_{AB}$ composed by two subsystems $\mathcal{S}_A$ and $\mathcal{S}_B$ described by their respective Hilbert space $\mathcal{H}_A \cong \mathbb{C}^{d_A}$ and $\mathcal{H}_B \cong \mathbb{C}^{d_B}$. We shall associate to the physical system $\mathcal{S}_{AB}$ the total Hilbert space $\mathcal{H}_{AB}$ which is given by the tensor product of $\mathcal{H}_A$ and $\mathcal{H}_B$ and we have $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

[4] In this thesis, we use the notation $[k]$ for the set $\{1, \cdots, k\}$.

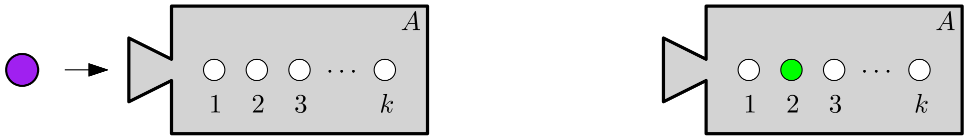


Figure 2.1: Diagrammatic representation of a quantum measurement apparatus. The device has an input canal and a set of k LEDs which will turn on when the corresponding outcome is achieved. After the measurement is performed, the particle is destroyed, and the apparatus displays the classical outcome (here, 2).

2.3.1 Pure states

One of the simplest examples that we can provide is that of two qubits. Let the first subsystem $\mathcal{S}_A$ be described by $\mathcal{H}_A \cong \mathbb{C}^2$ and the second subsystem $\mathcal{S}_B$ described $\mathcal{H}_B \cong \mathbb{C}^2$. Hence, the total Hilbert space associated to the system $\mathcal{S}_{AB}$ is given by $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^4$.

The tensor product structure in quantum mechanics induces two important fundamental classes of quantum states, known as *separable states* and *entangled states*. To illustrate that, let us consider the following two quantum states $|\psi\rangle$ and $|\varphi\rangle$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ given by

$$|\psi\rangle := \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \quad \text{and} \quad |\varphi\rangle := \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle).$$

where we have used $|i,j\rangle$ instead of $|i\rangle \otimes |j\rangle$. Note that the state $|\psi\rangle$ is separable: it is written as the tensor product. We remark that the state $|\varphi\rangle$ cannot be written as a $|\varphi_A\rangle \otimes |\varphi_B\rangle$ with $|\varphi_A\rangle \in \mathcal{H}_\mathcal{A}$ and $|\varphi_B\rangle \in \mathcal{H}_\mathcal{B}$. In general, we say that a state, that cannot be written as a tensor product of two states in each subsystem is *entangled*, contrary to those that can be written as the tensor product of the states of each subsystem which are called *separable state* (state like the quantum state $|\psi\rangle$).

2.3.2 Mixed states

As we have seen, density matrices can encode all the information about a physical system. In this more general setting, we can also describe a composite system, where we can imagine that during some experiments we want to measure a physical observable in a subregion of the total quantum system and the other part is inaccessible. For that, we should consider as before, a total system $\mathcal{S}_{AB}$ where the total Hilbert space is $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The quantum state of the total system is given by ρ_{AB}, assuming that in an experimental setting, we only have access to the physical subsystem $\mathcal{S}_A$, and we want to perform the measurement $A \in \mathcal{B}(\mathcal{H}_A)$. This operation from a mathematical point of view can be done by a *partial trace*.

Definition 2.3.1. *Let the finite dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Let the quantum state ρ_{AB} in $\mathcal{H}_{AB}$ and the observable $A = A^* \in \mathcal{B}(\mathcal{H}_A)$. The measurement of A on the subsystem $\mathcal{S}_A$ is on $\sigma_A \in \mathcal{M}_{d_A}^{1,+}$ defined by:*

$$\mathrm{Tr}[A\,\sigma_A] := \mathrm{Tr}_A[A\,\mathrm{Tr}_B(\rho_{AB})] = \mathrm{Tr}_{AB}[\rho_{AB}(A \otimes I_B)].$$

where $\sigma_A = \mathrm{Tr}_A[\rho_{AB}]$, $\mathrm{Tr}_{AB}[\cdot]$ is the trace over the total Hilbert space $\mathcal{H}_{AB}$, and $\mathrm{Tr}_A[\cdot]$,$\mathrm{Tr}_B[\cdot]$ is the trace over their respective Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$.

As before for pure states, the tensor product generates an important class of quantum states. We shall distinguish two classes of quantum states in a composite system $\mathcal{S}_{AB}$. We have *entangled state* and the *separable state*, for that let ρ_A quantum state in $\mathcal{H}_A$, ρ_B in $\mathcal{H}_B$, and the total quantum state ρ_{AB} in $\mathcal{H}_{AB}$. We have the following definition that establishes the difference between the *separable* and the *entangled* ones.

Definition 2.3.2. *A quantum state* $\rho_{AB} \in \mathcal{M}^{1,+}_{d_A d_B}$ *is said to be:*

- Product*, if* $\rho_{AB} = \rho_A \otimes \rho_B$*.*
- Separable*, if it is a* convex combination of product states*:*

$$\rho_{AB} = \sum_x p_x \, \rho_A^x \otimes \rho_B^x$$

with $p_x \geq 0$ *and* $\sum_x p_x = 1$*.*

- Entangled*, if is not separable.*

Above d_A *and* d_B *are the dimensions of the Hilbert spaces* $\mathcal{H}_A$ *and* $\mathcal{H}_B$ *respectively.*

The tensor product structure in quantum mechanics provides several important phenomena that distinguish completely the classical reality from the quantum one. Such phenomena are known as *nonlocal effects* in the sense if we create a pair of the entangled state and if we perform a measurement on one of the particles the outcome result on the other is completely determined. We will elaborate in Chapter 4 in great detail on the role played by the tensor product in the nonlocality context in general. Also, we shall see in the Subsection 2.5.2 how different notions of tensor products allow for different classes of states in GPTs.

2.4 Quantum Channels

Quantum channels are a very fundamental mathematical tool in quantum information theory. From a physical point of view, they represent the possible operations that we can perform on a physical system, generalizing the arbitrary dynamics of closed quantum systems from Postulate 2.1.3. From a mathematical point of view, such an operation is a subset of a more general class of maps known as *completely positive maps*. In the following, we shall give the definition of completely positive maps, which moreover, if they are also trace-preserving, are known as *quantum channels*. We shall give a brief application of the quantum channels that makes the quantum theory intrinsically different from the classical theory: the impossibility of perfect quantum cloning. At the end of this subsection, we shall introduce the dual maps of completely positive maps which can be understood physically as evolution map in the dual picture (Heisenberg picture).

One can say naively that the notion of completely positive maps is just a generalization of that of positive maps. They provide a very rich formalism from structural and mathematical points of view. Before giving the definition of the completely positive maps and quantum channels, we shall recall the definition of unital and positive maps.

Definition 2.4.1. *A* linear map $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_D(\mathbb{C})$ *is said to be* unital *if it satisfies the following condition*

$$\Phi(I_d) = I_D.$$

Definition 2.4.2. *A* linear map $\Phi(\cdot) : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_D(\mathbb{C})$ *is said to be positive if it satisfies the following property:*

$$X \in \mathcal{M}_d(\mathbb{C}) \quad , \quad X \geq 0 \implies \Phi(X) \geq 0.$$

We recall that $X \geq 0$ if $\sigma(X)$[5] $\subseteq [0, \infty[$. Another equivalent way of defining $X \geq 0 \iff \exists Y \in \mathcal{M}_d(\mathbb{C})$, such that $X = Y^*Y$. Such a matrix X is called positive semidefinite.

Now we are ready to introduce *complete positive maps*, which can be understood as a generalization of positive maps. If we add a trace preservation condition, the induced maps are known as quantum channels.

[5]We recall $\sigma(X)$ is the *spectrum of* X defined as the set of complex numbers z such that $X - zI$ is not invertible.

Definition 2.4.3. *A linear map $\Phi(\cdot) : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_D(\mathbb{C})$ is called* completely positive *if for all $K \geq 1$ and $X \in \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_K(\mathbb{C})$, we have*

$$X \geq 0 \implies [\Phi \otimes \mathrm{id}_K](X) \geq 0,$$

where id_K denotes the identity map on $\mathcal{M}_K(\mathbb{C})$. In other words, Φ is completely positive if $\forall K \geq 1$ the map $\Phi \otimes \mathrm{Id}_K$ is positive.

Definition 2.4.4. *The completely positive map $\Phi(\cdot)$, is a quantumm channel if it is trace preserving:*

$$\forall Y \in \mathcal{M}_d(\mathbb{C}), \qquad \operatorname{Tr} \Phi(Y) = \operatorname{Tr} Y.$$

Actually, there exists an equivalent way of describing quantum channels, that not required the use of the tensor product structure, and we have the following theorem known as the *Choi decomposition* that was first derived by Choi in [21].

Theorem 2.4.5. *[22, Theorem 2.21] A linear map $\Phi(\cdot) : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ is a quantum channel if only if $\exists \{L_i\}_{i=1}^k \subset \mathcal{M}_d(\mathbb{C})$ satisfying the following conditions:*

$$\forall X \quad , \quad \Phi(X) = \sum_{i=1}^{k} L_i X L_i^*,$$

and

$$\sum_{i=1}^{k} L_i^* L_i = I_d.$$

All that we have seen from the beginning of this section, are the possible operation that we can act with quantum states. Those types of operations can be interpreted as the possible transformation in the Schrödinger representation. As we know the existence of a dual representation, the Heisenberg one deals more with the observable and fixed quantum states. We can ask by this duality, what are the possible transformation in the Heisenberg picture?

For that, in general let $X \in \mathcal{M}_d$ and $Y \in \mathcal{M}_D$ and a linear map $\Phi(\cdot) : \mathcal{M}_d \to \mathcal{M}_D$ where

$$\langle Y, \Phi(X) \rangle_{HS} = \langle \Phi^*(Y), X \rangle_{HS}.$$

with $\Phi^*(\cdot) : \mathcal{M}_D \to \mathcal{M}_d$ is the dual map of $\Phi(\cdot)$.

This duality allows passing from the Schrödinger to the Heisenberg representation. If the map $\Phi(\cdot)$ is *trace preserving* it induces a *unital dual map* $\Phi^*(\cdot)$ while if it is *completely positive* it induces a *complete positive map* [10, Section 4.1.2]. Hence if one acts on a quantum state by *completley positive trace-preserving map* it induces a *completely positive unital map.* As an application of quantum channels, we turn to one of the major differences between the classical and the quantum theory: *a quantum state cannot be copied*, while for classical states, we can make as many copies as we want. More precisely we have the *no-cloning theorem* which can be formulated as follows:

Theorem 2.4.6. *[23] For any number of clones $g \geq 2$, there is no quantum channel $\Phi : \mathcal{M}_d \to \mathcal{M}_d^{\otimes g}$ with the property that*

$$\forall \rho \in \mathcal{M}_d^{1,+}, \forall j \in [g], \qquad \operatorname{Tr}_{[g]\setminus\{j\}} \Phi(\rho) = \rho.$$

where $\operatorname{Tr}_{[g]\setminus j}[\cdot]$ is the trace over all the elements excepts the j one.

In order to avoid this problem, several works have been investigated, the situation where instead of requiring as an output perfect clones, one can obtain copies with some noise and it is known as the *approximate quantum cloning* that we will introduce in Chapter 5.

2.5 Different approaches to quantum theory

The development of the quantum theory has led to different approaches to the theory. In this section, we shall introduce only two different approaches to quantum theory. The first one, the oldest one, is the *algebraic quantum mechanics*. This formulation was established with the birth of the quantum theory, see [24] for a historical overview. This approach is also used in quantum field theory and quantum statistical mechanics [16, 18].

The second approach that we shall mention is the *general probabilistic theory* [25]. This approach was developed in order to understand quantum theory from an axiomatic point of view. Its goal is to describe with a minimal amount of axioms a physical theory, including finite-dimensional quantum theory as a special case, but also allowing for more general and exotic physical theories.

2.5.1 Algebraic approach

The algebraic approach to quantum theory is based on the C^*-algebraic framework, see for example [16–18] and the lectures [26, 27]; in this approach, the Hilbert space appears in the theory as secondary. The main objects used in this framework are operators and their representations. Generally, the algebraic approach becomes relevant when one uses an infinite number of degrees of freedom (see eg [18]) that appear when one studies quantum field theory or quantum statistical physics, which are far beyond this thesis. However, we will see in Chapter 4 that a famous conjecture in operator algebras was recently solved by using techniques and concepts from nonlocal games.

Before introducing C^*-algebras, we recall the definition of a *Banach algebra*. A Banach algebra is a *Banach space*[6] with an operation $\circ$ satisfying the following condition $\forall x, y \in X$, $\|x \circ y\| \leq \|x\|\|y\|$. We note a *Banach algebra* by $\mathcal{A}$. An *involution* is a map $* : \mathcal{A} \to \mathcal{A}$ satisfying the following properties $\forall x, y \in \mathcal{A}, \forall \alpha \in \mathbb{C}$, $(x+y)^* = x^* + y^*$, $(\alpha x)^* = \bar{\alpha} x^*$, $(xy)^* = y^* x^*$[7]. We call a Banach algebra equipped with an evolution operation as *involutive Banach algebra*.

Definition 2.5.1. *A* C^*-algebra[8] *is an involutive Banach algebra satisfying the following property*

$$\forall x \in \mathcal{A} \quad \|x^* x\| = \|x\|^2.$$

We shall recall a *linear functional* $\omega(\cdot)$ on a C^*-algebra $\mathcal{A}$ is a map $\omega(\cdot) : \mathcal{A} \to \mathbb{C}$ satisfying the following properties $x, y \in \mathcal{A}$, $\lambda \in \mathbb{C}$, $\omega(x+y) = \omega(x) + \omega(y)$, $\omega(\lambda x) = \lambda \omega(x)$, $\omega(x^*) = \overline{\omega(x)}$.

Definition 2.5.2. *A* state *is a linear functional satisfying*

$$\forall x \in \mathcal{A} \quad \omega(x^* x) \geq 0.$$

A state is said to be faithful *if* $\omega(x^* x) = 0 \implies x = 0$.

A *representation* $\pi_\omega(.) : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$, is an *involutive homomorphism* satisfying the following properties $\pi_\omega(xy) = \pi_\omega(x)\pi_\omega(y)$, $\pi_\omega(x+y) = \pi_\omega(x) + \pi_\omega(y)$ and $\pi_\omega(x^*) = \pi_\omega(x)^*$ on a Hilbert space $\mathcal{H}_\omega$[9].

The GNS theorem assures that for each state $\omega(\cdot)$ one can associate to it a representation on a Hilbert space $\mathcal{H}_\omega$, a representation of the algebra $\mathcal{A}$ on the bounded operators of $\mathcal{H}_\omega$ and a *unique vector* $\Omega_\omega \in \mathcal{H}_\omega$.

[6] Banach space, is a vector space X (not necessarily finite) on $\mathbb{C}$ endowed with a norm $\|x\| < \infty$ for all $x \in X$.

[7] We have used the notation xy instead of $x \circ y$.

[8] We shall only work with *unital* C^*-algebras i.e., $\mathrm{I} \in \mathcal{A} \implies \omega(\mathrm{I}) = 1$.

[9] The Hilbert space $\mathcal{H}_\omega$ is not necessarily finite dimensional.

Theorem 2.5.3. *(GNS [10] theorem) [17, Theorem 9.14 Chapter 1]*

Given a C^*-algebra $\mathcal{A}$ and a state ω, then there exist a unique *representation* $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ *such that $\forall A \in \mathcal{A}$ we have*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle.$$

We call the triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ a representation *induced* by ω. Hence, there is a natural correspondence between C^*-algebras and bounded operators. From this theorem, we can observe that the Hilbert space plays a secondary role not a fundamental one. One last class of algebraic objects that play a fundamental role in quantum mechanics and quantum field theory [11] are the *von Neumann algebras*.

Definition 2.5.4. *[17, Definition 3.2, Chapter 2] A von Neumann algebra is a sub-algebra $\tilde{\mathcal{A}} \subseteq \mathcal{B}(\mathcal{H})$ which is equal to its bicommutant:*

$$\tilde{\mathcal{A}}'' = \tilde{\mathcal{A}}.$$

Above the *commutant* of $\tilde{\mathcal{A}}$ is defined by $\tilde{\mathcal{A}}' := \{C \in \mathcal{B}(\mathcal{H}) | \forall A \in \tilde{\mathcal{A}} : \; AC = CA\}$ and the bicommutant is $(\tilde{\mathcal{A}}')'$. A von Neumann algebra is called a *factor* if the center $\mathcal{C}$ of $\tilde{\mathcal{A}}$ contains only multiples of the identity $\mathcal{C} := \tilde{\mathcal{A}} \cap \tilde{\mathcal{A}}' = \mathbb{C}I$.

2.5.2 Generalised probabilistic theories

A *generalized probabilistic theory* (or simply GPT), is an attempt to understand what makes quantum theory the quantum theory. The aim of this approach is to define all the concepts used in quantum theory with a minimal amount of axioms. It turns out that the *finite* dimensional quantum theory is an example of a GPT. In the following, we shall briefly give a definition of *state space*, *effect*, and the *tensor product structure* of a GPT; for more details and a complete introduction to the topic, see [20] and the references therein. The motivation of this part will become relevant in Chapter 5 with the notion of compatibility tensor norms, which was first introduced in the context of GPTs. The starting point of the work [2], was the observation of reducing the compatibility tensor norm from GPT [28] in the context of quantum mechanics. In turn, thsis gives a natural framework that unifies quantum nonlocality with the incompatibility of quantum measurement.

Definition 2.5.5. *In a GPT, a* state space *$\mathcal{K}$ is a:*

- *Set of points which is:*
- *Convex,*
- *Bounded,*
- *Subset of a real, finite-dimensional vector space with Euclidean topology.*

If we want to model the outcomes of a given measurement on a given state in $\mathcal{K}$, we use the notion of an effect.

Definition 2.5.6. *An* effect *f is defined as an affine function $f : \mathcal{K} \to [0, 1]$*

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y).$$

$\forall x, y \in \mathcal{K}$ and $\lambda \in [0, 1]$.

We denote the set of all effects on $\mathcal{K}$ by $A(\mathcal{K})$; this set is known as effect algebra.

[10] GNS, stands for Gelfand, Naimark, and Segal.

[11] In quantum theory they are reduced to bounded operators on Hilbert spaces. However, in quantum field theory and quantum statistical physics, they become relevant when one considers infinite degrees of freedom.

Actually, the effect algebra $A(\mathcal{K})$ has a nice ordering structure, see [20, Lemma 3.12]. From this lemma, we define the set analogous to the measurement used in the usual quantum theory in the sense that the outcomes are in the set $[0,1]$, which motivates the following proposition

Proposition 2.5.7. *[20] The measurement set $E(\mathcal{K})$ in GPT is given by:*

$$E(\mathcal{K}) = \{f \in A(\mathcal{K}) : 0 \leq f \leq \mathrm{I}_{\mathcal{K}}\}$$

In the above definitions, we have described the notion of state and the notion of effects. Actually, all the concepts used in the quantum theory can also be defined and analyzed in the setting of GPT, such as the notion of the channels, the compatibility of effects [29] and the compatibility of channels [25]. As we have seen in the quantum theory to describe a bipartite system one needs the tensor product structure. In GPTs, we can also consider the case of a bipartite system composed from two state spaces $\mathcal{K}_A$ and respectively $\mathcal{K}_B$. It turns out that if we want to consider both spaces, a tensor product structure is needed where the bipartite system $\mathcal{K}_{AB} = \mathcal{K}_A \otimes \mathcal{K}_B$ should be a valid state space in the sense of Definition 2.5.5. The composite state space $\mathcal{K}_{AB}$ lies between a *maximal* and a *minimal* tensor product of $\mathcal{K}_A$ and $\mathcal{K}_B$. More precisely we have

$$\mathcal{K}_A \otimes_{min} \mathcal{K}_B \subseteq \mathcal{K}_A \otimes \mathcal{K}_B \subseteq \mathcal{K}_A \otimes_{max} \mathcal{K}_B$$

with the minimal tensor product of $\mathcal{K}_A$ and $\mathcal{K}_B$ is given by

$$\mathcal{K}_A \otimes_{min} \mathcal{K}_B := \mathrm{conv}(\{x_A \otimes x_B : x_A \in \mathcal{K}_A, x_B \in \mathcal{K}_B\}),$$

and the maximal tensor product is given by

$$\mathcal{K}_A \otimes_{max} \mathcal{K}_B := \{\varphi \in A(\mathcal{K}_A)^* \otimes A(\mathcal{K}_B)^* : \langle \varphi, f_A \otimes f_B \rangle \geq 0, \forall f_A \in E(\mathcal{K}_A), \forall f_B \in E(\mathcal{K}_B), \langle \varphi, I_{\mathcal{K}_A} \otimes I_{\mathcal{K}_B} \rangle = 1\}.$$

We shall mention that the tensor product in the quantum theory is a particular one:

$$\mathrm{QM}_A \otimes_{min} \mathrm{QM}_B \subset \mathrm{QM}_{AB} \subset \mathrm{QM}_A \otimes_{max} \mathrm{QM}_B.$$

The notions of minimal and maximal tensor products of GPTs are related also to the injective and projective tensor norms that we shall introduce in Subsections 3.2.2 and 3.2.1. One important result that was recently shown in [30] is that the equality between the maximal and the minimal tensor product hold if and only if one of the spaces $\mathcal{K}_A$ or $\mathcal{K}_B$ is a *simplex*.

Chapter 3

Tensor norms on Banach spaces

In this chapter, we will give basic definitions and examples of finite dimensional Banach spaces, along with some fundamental theorems, for a more general introduction to the theory, see [31, 32]. We will recall the basic notion of tensor product in Section 3.1, and we shall expose briefly the theory of the tensor product of Banach spaces in Section 3.2, where we shall only focus on the finite-dimensional spaces, see for example [33] for an introduction. We will introduce the *projective tensor norm* and *injective tensor norm* and we will end with a general definition of *tensor norms on Banach spaces* [34]. The theory of tensor products on Banach spaces will play an important role in the nonlocality context in Chapter 4 and will be the key to understanding the link between incompatibility and nonlocality in Chapter 6.

3.1 Banach spaces

We will start this chapter by fixing the notations. We recall that a finite-dimensional *Banach space* is a vector space X, endowed with a norm $\|\cdot\|_X$, and we write $(X, \|\cdot\|_X)$. In the following we will use X, Y, Z to denote different Banach spaces with there respective norms $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$. Let us recall some basic Banach spaces that will be used in this manuscript.

We note the space of linear maps from the vector space X to Y by $\mathcal{L}(X, Y)$. In particular if we consider two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ we shall denote the space of all linear maps from X to Y by $\mathcal{L}(X, Y)$ and we have

$$\varphi \in \mathcal{L}(X, Y) \quad , \quad \varphi : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y).$$

Remark 3.1.1. *We will use the usual convention of writing $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. We recall that the dual vector space X^* can be identified with $\mathcal{L}(X, \mathbb{C})$. Its elements are known as one forms.*

Actually, we can endow the space of a linear map from X to Y, with a norm structure defined by:

$$\varphi \in \mathcal{L}(X, Y) \quad , \quad \|\varphi\|_{X \to Y} := \sup\{\, \|\varphi(x)\|_Y,\ \|x\|_X \leq 1\}.$$

Hence the space of linear maps forms a new Banach space with the norm above.

In particular, the dual space of forms $X^* = \mathcal{L}(X, \mathbb{C})$ is a Banach space endowed with the following norm:

$$\|\varphi\|_{X^*} := \sup\{\, |\varphi(x)|,\ \|x\|_X \leq 1\}. \tag{3.1}$$

We note the dual normed space as $(X^*, \|\cdot\|_{X^*})$.

3.1.1 Examples of Banach spaces

We shall introduce now some well-known Banach spaces such as $\ell_p^N(\mathbb{R})$ spaces and their non-commutative analog $\mathcal{S}_p^N(\mathbb{R})$ spaces defined on the vector space of matrices. We shall also recall the *Hölder theorem.*

We recall that $\ell_p^N(\mathbb{R})$ are vector spaces $\mathbb{R}^N$ endowed with the norm $\|\cdot\|_p$ for given p satisfying $1 \leq p \leq +\infty$.

Definition 3.1.2. *The Banach space $\ell_p^N(\mathbb{R})$ defined by $\ell_p^N(\mathbb{R}) = (\mathbb{R}^N, \|\cdot\|_p)$ where the norm $\|\cdot\|_p$ is defined as follow:*

$$\|\cdot\|_p : \mathbb{R}^N \to \mathbb{R}^+,$$

$$x \to \|x\|_p := \begin{cases} \left(\sum_{i=1}^N |x_i|^p\right)^{\frac{1}{p}} & \textit{if} \quad 1 \leq p < \infty \\ \\ \sup\left\{|x_i|, i \in \{1, \cdots, N\}\right\} & \textit{if} \quad p = \infty. \end{cases}$$

Theorem 3.1.3. *(Hölder theorem [35]) Let $x \in \ell_p^N(\mathbb{R})$ and $y \in \ell_q^N(\mathbb{R})$ for given integer p and q satisfying $1 \leq p, q \leq +\infty$ satisfying the following condition*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$|\langle x, y\rangle| \leq \sum_{i=1}^N |x_i\, y_i| \leq \|x\|_p \, \|y\|_q.$$

Remark 3.1.4. *The above inequality reduces to* Cauchy-Schwarz inequality *for $p = q = 2$.*

By the Hölder theorem we can identify the dual space for a given $\ell_p^N(\mathbb{R})$ (see [35]):

$$(\ell_p^N(\mathbb{R}))^* = (\mathbb{R}^N, \|\cdot\|_p)^* = \ell_q^N(\mathbb{R}) = (\mathbb{R}^N, \|\cdot\|_q),$$

for given p and q satisfying $1 \leq p, q \leq +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Example 3.1.5. *The two examples that we shall mostly be interested in are $\ell_1^N(\mathbb{R})$ and $\ell_\infty^N(\mathbb{R})$ (see for example [35]).*

- *The space $\ell_1^N(\mathbb{R}) = (\mathbb{R}^N, \|\cdot\|_1)$, where the norm is given by*

$$\forall x \in \mathbb{R}^N \quad , \quad \|x\|_1 = \sum_{i=1}^N |x_i|.$$

- *The space $\ell_\infty^N(\mathbb{R}) = (\mathbb{R}^N, \|\cdot\|_\infty)$, where the norm is given by*

$$\forall x \in \mathbb{R}^N \quad , \quad \|x\|_\infty = \max_i |x_i|.$$

Remark 3.1.6. *As we are working in finite-dimensional vector spaces, all the norms are equivalent, given rise to the same topology.*

In the following, we will introduce the noncommutative analogs of $\ell_p^N(\mathbb{R})$ spaces. We can endow the space of complex matrices $\mathcal{M}_N(\mathbb{C})$ with a norm. We will introduce the analog of the Hölder inequality in the non-commutative setting, and some examples to illustrate the structure of such spaces.

Let $\mathcal{M}_N(\mathbb{C})$ be the space of matrices, we can endow it with a norm known as the *Schattern $p-$norm* and denote the Banach space by $\mathcal{S}_p^N(\mathbb{C}) := (\mathcal{M}_N(\mathbb{C}), \|\cdot\|_p)$.

Definition 3.1.7. *The Banach space* $\mathcal{S}_p^N(\mathbb{C}) = (\mathcal{M}_N(\mathbb{C}), \|\cdot\|_p)$ *for* $1 \le p \le +\infty$,

$$\|\cdot\|_p : \mathcal{M}_N(\mathbb{C}) \to \mathbb{R}^+,$$

$$M \to \|M\|_p := \begin{cases} \Big(\operatorname{Tr}|M|^p\Big)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty. \\ \\ \sup\Big\{\|M \cdot x\|,\ \|x\| \le 1\Big\} & \text{if} \quad p = \infty. \end{cases}$$

where $|M| := \sqrt{(M^*M)}$ *and* $\|\cdot\|$ *is the euclidean norm of vector in* $\mathbb{C}^N$*.*

We have the following non-commutative version of the *Hölder theorem.*

Theorem 3.1.8. *(Hölder theorem [22, Proposition 1.17]) Let* $M \in \mathcal{S}_p^N(\mathbb{C})$ *and* $N \in \mathcal{S}_q^N(\mathbb{C})$ *for given integer* p *and* q *satisfying* $1 \le p, q \le +\infty$ *satisfying the following condition*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$|\langle M, N\rangle| = |\operatorname{Tr}(M^*N)| \le \|M\|_p\, \|N\|_q.$$

Remark 3.1.9. *The above inequality reduces to* Cauchy Schwarz inequality *for* $p = q = 2$ *with the* Hilbert-Schmidt *scalar product* $\langle M, N\rangle := \operatorname{Tr}(M^*N)$*.*

By the Hölder theorem, we can identify the dual space for a given non-commutative $\mathcal{S}_p^N(\mathbb{C})$ space. Hence we have the following duality between Banach spaces (see [22, Corollary 1.18]):

$$(\mathcal{S}_p^N(\mathbb{C}))^* = (\mathcal{M}_N(\mathbb{C}), \|\cdot\|_p)^* = \mathcal{S}_q^N(\mathbb{C}) = (\mathcal{M}_N(\mathbb{C}), \|\cdot\|_q),$$

for given p and q satisfying $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Example 3.1.10. *The two example that we shall consider are* $\mathcal{S}_1^N(\mathbb{C})$ *and* $\mathcal{S}_\infty^N(\mathbb{C})$*.*

- *The space* $\mathcal{S}_1^N(\mathbb{C}) = (\mathcal{M}_N(\mathbb{C}), \|\cdot\|_1)$ *where the norm is given by*

$$\forall M \in \mathcal{M}_N(\mathbb{C}) \quad , \quad \|M\|_1 = \operatorname{Tr}|M|.$$

- *The space* $\mathcal{S}_\infty^N(\mathbb{C}) = (\mathcal{M}_N(\mathbb{C}), \|\cdot\|_\infty)$ *where the norm is the natural operator norm given by*

$$\forall M \in \mathcal{M}_N(\mathbb{C}) \quad , \quad \|M\|_\infty = \sup_{\|x\| \le 1} \|M \cdot x\|,$$

where $\|\cdot\| = \|\cdot\|_2$ *is the euclidean norm of vectors in* $\mathbb{C}^N$*.*

3.2 Tensor products of Banach spaces

In this section, we shall introduce the tensor product structure on Banach spaces. To fix the notations we shall recall the definition of tensor product structure in general. The main question that we shall answer in this subsection is the following: *given two Banach spaces (or more). What is a natural norm to consider on the tensor product of their vector spaces?*

We shall introduce the concept of *tensor norm* which is just a norm structure that we put on the tensor product to make it a Banach space. We shall distinguish two types of norms known as the *projective tensor norm* and *the injective tensor norm.* We shall recall the definitions of the two norms, with some of their properties and some examples. We will introduce the general main definition of a *tensor norm*, where we will see that the projective and the injective tensor norm play the role of the maximal and the minimal tensor norm that we can put on a given tensor product space.

Let X, Y, and Z be three finite dimensional vector spaces. A *bilinear map* B is a map from $X \times Y$ to Z:

$$B : X \times Y \to Z.$$

satisfying the following properties:

$$B(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 B(x_1, y) + \lambda_2 B(x_2, y).$$
$$B(x, \lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 B(x, y_1) + \lambda_2 B(x, y_2).$$

$\forall \lambda_1, \lambda_2 \in \mathbb{C}$, $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$.

We shall note the space of *biliniear maps* from $X \times Y$ to Z as $\mathcal{B}(X \times Y, Z)$. In particular if $Z = \mathbb{C}$, we shall note the space of *bilineear forms* as $\mathcal{B}(X \times Y)$ instead of $\mathcal{B}(X \times Y, \mathbb{C})$.

The tensor product of two vector spaces X and Y is the space of *linear functionals* on $\mathcal{B}(X \times Y)$, and we denote it by $X \otimes Y$. The resulting tensor product space is defined *up to isomorphism*, and we have the following identification, $\forall x \in X$, $\forall y \in Y$ we have:

$$(x \otimes y)(B) = B(x, y).$$

for all $B \in \mathcal{B}(X \times Y)$.

With the construction above, we shall recall that we have a well-known identification between bilinear maps and linear maps in finite-dimensional spaces.

$$\mathcal{L}(X, Y^*) = \mathcal{B}(X \times Y) = (X \otimes Y)^*.$$

Generally, a representation of a tensor is *not unique*, we say a tensor $T \in X \otimes Y$ is *simple* if $T = x \otimes y$ where $x \in X$ and $y \in Y$, more generally a simple tensor $T \in \bigotimes_{i=1}^{M} X_i$ decomposes as $T = x_1 \otimes \cdots \otimes x_M$ where $x_i \in X_i$.

Definition 3.2.1. *The tensor rank or a rank of a tensor $T \in X \otimes Y$ is defined as the minimum integer r, where the tensor is represented as the sum of simple tensors in $X \otimes Y$:*

$$T = \sum_{i=1}^{r} x_i \otimes y_i.$$

Remark 3.2.2. *In particular, a tensor of rank one is a simple tensor.*

Remark 3.2.3. *The definition 3.2.1 can be generalised for tensor $T \in \bigotimes_{i=1}^{M} X_i$, where the rank of T is given by the minimum integer r such that T decomposes as:*

$$T = \sum_{i=1}^{r} x_i^1 \otimes x_i^2 \otimes \cdots \otimes x_i^M.$$

where $x_i^j \in X_j$.

Let the following finite dimensional Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$. One can define the space of *bounded bilinear maps* as the space of bilinear maps $\mathcal{B}(X \times Y, Z)$ endowed with the following bounded norm

$$B \in \mathcal{B}(X \times Y, Z) \quad , \quad \|B\| := \sup\{\|B(x, y)\|_Z \,;\, \|x\|_X \leq 1 \,,\, \|y\|_Y \leq 1\}.$$

Given two finite-dimensional Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, if we construct the tensor product of $X \otimes Y$ *which norm we shall put on tensor structure of $X \otimes Y$?* To answer this question we shall introduce in the following *the projective and the injective tensor norm* and *the general notion of tensor norm.*

3.2.1 The projective tensor norm

In the following, we shall give a brief introduction to *projective tensor norm*, which is a norm structure on the tensor product of two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$.

Definition 3.2.4. *Given two finite-dimensional Banach spaces X and Y with their respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and $u \in X \otimes Y$, we define the* projective tensor norm *of u as:*

$$\|u\|_{X\otimes_\pi Y} := \inf \left\{ \sum_{i=1}^{N} \|x_i\|_X \|y_i\|_Y : u = \sum_{i=1}^{N} x_i \otimes y_i \right\},$$

where the infimum is taken over all the decomposition of $u = \sum_{i=1}^{N} x_i \otimes y_i$ where N is a finite but arbitrary integer.

Such a norm endows the tensor product $X \otimes Y$ with a Banach space structure. We shall denote the induced space by $X \otimes_\pi Y$, which is the tensor product of $X \otimes Y$ endowed with the projective norm. We note the Banach space $X \otimes_\pi Y := \Big(X \otimes Y, \|\cdot\|_{X\otimes_\pi Y}\Big)$.

One can check easily that if we consider a simple tensor its projective norm is given by

$$u = x \otimes y \in X \otimes Y \quad \Longrightarrow \quad \|u\|_{X\otimes_\pi Y} = \|x\|_X \|y\|_Y.$$

Remark 3.2.5. *The definition above can be extended for more than two Banach spaces. To illustrate that, let us consider M Banach spaces $(X_i, \|\cdot\|_{X_i})$, for $i \in \{1 \cdots M\}$. We can endow the tensor product of all the M Banach spaces by the projective norm defined as*

$$u \in \bigotimes_{i=1}^{M} X_i \quad , \quad \|u\|_\pi := \inf \Big\{ \sum_{k=1}^{r} \|x_k^1\| \cdots \|x_k^M\| : r \in \mathbb{N}\,,\, x_k^i \in X_i\,,\, u = \sum_{k=1}^{r} x_k^1 \otimes \cdots \otimes x_k^M \Big\}.$$

where we have used the shorthand notation $\|\cdot\|_\pi$ instead of $\|\cdot\|_{X_1\otimes_\pi \cdots \otimes_\pi X_M}$.

The projective tensor norm satisfies the following fundamental property known as the *metric mapping property.*

Definition 3.2.6. *[33] Let the linear maps $T \in \mathcal{L}(X, Z)$ and $S \in \mathcal{L}(Y, W)$ where $(X, \|\cdot\|_X)\,, (Y, \|\cdot\|_Y)\,, (Z, \|\cdot\|_Z)\,, (W, \|\cdot\|_W)$ are Banach spaces, we say the induced norm on $X \otimes Y$ satisfies the metric mapping property if for all bilinear maps $T \otimes S$ the following holds:*

$$\|T \otimes S\| \leq \|T\|\, \|S\|.$$

Lemma 3.2.7. *Let the linear maps $T \in \mathcal{L}(X, Z)$ and $S \in \mathcal{L}(Y, W)$, and the Banach spaces $X \otimes_\pi Y$ and $Z \otimes_\pi W$. The projective norm satisfies the metric mapping property. More precisely we have:*

$$\|T \otimes S : X \otimes_\pi Y \to Z \otimes_\pi W\| \leq \|T\|\|S\|.$$

where $\|T \otimes S : X \otimes_\pi Y \to Z \otimes_\pi W\| := \sup_{\|u\|_{X\otimes_\pi Y} \leq 1} \|(T \otimes S)(u)\|$.

Proof. The linear map $T \otimes S : \; X \otimes_\pi Y \to Z \otimes_\pi W$ is defined by the canonical extension of $T \otimes S : \; X \otimes Y \to Z \otimes W$ given by

$$\Big(T \otimes S\Big) \sum_{i=1}^{N} x_i \otimes y_i = \sum_{i=1}^{N} T(x_i) \otimes S(y_i).$$

Let $u = \sum_{i=1}^N x_i \otimes y_i \in X \otimes_\pi Y$ and the map $T \otimes S : X \otimes_\pi Y \to Z \otimes_\pi W$, then we have:

$$u = \sum_{i=1}^N x_i \otimes y_i \quad , \quad \|(T \otimes S)(u)\| = \Big\|\Big(T \otimes S\Big)\Big(\sum_{i=1}^N x_i \otimes y_i\Big)\Big\| = \Big\|\sum_{i=1}^N T(x_i) \otimes S(y_i)\Big\|$$
$$\leq \sum_{i=1}^N \|T(x_i) \otimes S(y_i)\| = \sum_{i=1}^N \|T(x_i)\|\|S(y_i)\|$$
$$\leq \|T\|\|S\| \sum_{i=1}^N \|x_i\|_X \|y_i\|_Y.$$

Take now the supremum over all $\|u\|_{X \otimes_\pi Y} \leq 1$, and use the definition of $\|T \otimes S : X \otimes_\pi Y \to Z \otimes_\pi W\|$ to show the *metric mapping property* holds. □

Example 3.2.8. *The simplest example that we can provide to illustrate the projective tensor norm is the following. Let $u \in \ell_2^N(\mathbb{R}) \otimes_\pi \ell_2^N(\mathbb{R})$ one can check easily that*

$$\|u\|_{\ell_2^N(\mathbb{R}) \otimes_\pi \ell_2^N(\mathbb{R})} = \mathrm{Tr}\,|U| = \|U\|_1$$

where U is the matrix representation of u on $\mathcal{M}_N(\mathbb{R}) \cong \mathbb{R}^N \otimes \mathbb{R}^N$. We have the following identification of Banach spaces:

$$\ell_2^N(\mathbb{R}) \otimes_\pi \ell_2^N(\mathbb{R}) = (\mathcal{M}_N(\mathbb{R}), \|\cdot\|_1).$$

Remark 3.2.9. *Generally is not easy to compute the projective tensor norms explicitly, concrete computations can be done only on some specific examples.*

Before ending this subsection, we shall answer the following question: *what is the dual space of the tensor product of two spaces endowed with a projective tensor norm ?*

To answer this question, let the following Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ and a bounded bilinear map $B \in \mathcal{B}(X \times Y, Z)$. By using the duality between bilinear maps and the tensor product structure, we have the following unique identification on the tensor product $X \otimes Y$.

Theorem 3.2.10. *[33] Let $B : X \times Y \to Z$ be a bounded bilinear mapping. Then there exists a unique operator $\tilde{B} : X \otimes_\pi Y \to Z$ satisfying $\tilde{B}(x \otimes y) = B(x, y)$, for every $x \in (X, \|\cdot\|_X)$ and $y \in (Y, \|\cdot\|_Y)$. Such identification is an* isometric isomorphism *between the Banach spaces $\mathcal{B}(X \times Y, Z)$ and $\mathcal{L}(X \otimes_\pi Y, Z)$.*

The theorem above ensures the following identification between bounded bilinear maps from $X \times Y$ to Z and linear maps from $X \otimes_\pi Y$ to Z:

$$\mathcal{B}(X \times Y, Z) = \mathcal{L}(X \otimes_\pi Y, Z).$$

In particular, if Z is a scalar field, we have the following identification of the space of bilinear forms on $X \times Y$ and dual of $X \otimes_\pi Y$:

$$\mathcal{B}(X \times Y) = \Big(X \otimes_\pi Y\Big)^*.$$

Such identification will lead us to another norm structure that we can put on the tensor product of two (or more) Banach spaces known as the *injective tensor norm*, which we will introduce in the following subsection.

3.2.2 The injective tensor norm

In the following, we shall give a brief introduction to *injective tensor norm*, which is another norm structure on the tensor product of two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$.

Definition 3.2.11. *Given two finite dimensional Banach spaces X and Y with their respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and $u \in X \otimes Y$, we define the* injective tensor norm *of u as:*

$$\|u\|_{X\otimes_\varepsilon Y} := \sup_{\|\lambda\|_{X^*}\,,\,\|\sigma\|_{Y^*}\leq 1} |\langle \lambda\otimes\sigma, u\rangle|.$$

where $\lambda \in X^*$ and $\sigma \in Y^*$ are linear forms.

Remark 3.2.12. *In the definition above, we have used the duality bracket: $\langle \lambda\otimes\sigma, \cdot\rangle$ should be understood as $(\lambda\otimes\sigma)(\cdot)$.*

Such a norm endows the tensor product vector space $X \otimes Y$ with a Banach space structure. We shall denote the induced space by $X \otimes_\varepsilon Y$ which is the tensor product of $X \otimes Y$ endowed with the injective norm, hence $X \otimes_\varepsilon Y := \Big(X \otimes Y, \|\cdot\|_{X\otimes_\varepsilon Y}\Big)$.

One can check easily if we consider a simple tensor its injective norm is given by the product of the norms of the corresponding factors

$$u = x\otimes y \in X\otimes Y \quad \Longrightarrow \quad \|u\|_{X\otimes_\varepsilon Y} = \|x\|_X\|y\|_Y.$$

Remark 3.2.13. *The definition above can be extended to more than two Banach spaces. To illustrate that, let us consider the M Banach spaces $(X_i, \|\cdot\|_{X_i})$, for $i \in \{1\cdots M\}$. We can endow the tensor product of the M Banach spaces by the injective norm defined as*

$$u \in \bigotimes_{i=1}^{M} X_i \quad , \quad \|u\|_\varepsilon := \sup\Big\{|\langle x^1\otimes\cdots\otimes x^M, u\rangle|;\; x^i\in X_i^*\,,\, \|x^i\|_{X_i^*}\leq 1\Big\}.$$

where we have used the shorthand notation $\|\cdot\|_\varepsilon$ instead of $\|\cdot\|_{X_1\otimes_\varepsilon\cdots\otimes_\varepsilon X_M}$.

Lemma 3.2.14. *Consider the linear maps $T \in \mathcal{L}(X, Z)$ and $S \in \mathcal{L}(Y, W)$, and the Banach spaces $X \otimes_\varepsilon Y$ and $Z \otimes_\varepsilon W$. The projective norm satisfies the metric mapping property. More precisely we have:*

$$\|T\otimes S : X\otimes_\varepsilon Y \to Z\otimes_\varepsilon W\| \leq \|T\|\|S\|.$$

where $\|T\otimes S : X\otimes_\varepsilon Y \to Z\otimes_\varepsilon W\| := \sup_{\|u\|_{X\otimes_\varepsilon Y}\leq 1} \|(T\otimes S)(u)\|$.

Proof. Let $u = \sum_{i=1}^N x_i\otimes y_i \in X\otimes_\varepsilon Y$. The injective norm of u satisfies the following inequality:

$$\|u\|_{X\otimes_\varepsilon Y} = \sup_{\|\lambda\|_{X^*},\|\sigma\|_{Y^*}\leq 1} |\langle\lambda\otimes\sigma, u\rangle| = \sup_{\|\lambda\|_{X^*},\|\sigma\|_{Y^*}\leq 1} \Big|\sum_{i=1}^N \lambda(x_i)\sigma(y_i)\Big| \leq \sum_{i=1}^N \|x_i\|_X\|y_i\|_Y.$$

By applying the bilinear map $T \otimes S$ on u we have:

$$u = \sum_{i=1}^N x_i\otimes y_i \quad , \quad \|(T\otimes S)(u)\| \leq \|T\|\|S\|\sum_{i=1}^N \|x_i\|_X\|y_i\|_Y$$

By taking the supremum over $\|u\|_{X\otimes_\varepsilon Y} \leq 1$ in the definition of $\|T\otimes S : X\otimes_\varepsilon Y \to Z\otimes_\varepsilon W\|$, we see that the metric mapping property holds. □

Example 3.2.15. *The simplest example that we can provide to illustrate the injective tensor norm is the following. For* $u \in \ell_2^N(\mathbb{R}) \otimes_\varepsilon \ell_2^N(\mathbb{R})$*, one can check easily that*

$$\|u\|_{\ell_2^N(\mathbb{R})\otimes_\varepsilon \ell_2^N(\mathbb{R})} = \sup_{\|\lambda\|_2\,,\,\|\sigma\|_2\leq 1} \langle\lambda|\, U\, |\sigma\rangle = \|U\|_\infty,$$

where U *is the matrix representation of* u *in* $\mathcal{M}_N(\mathbb{R})$*. We have the following identification of Banach spaces*

$$\ell_2^N(\mathbb{R}) \otimes_\varepsilon \ell_2^N(\mathbb{R}) = (\mathcal{M}_N(\mathbb{R}), \|\cdot\|_\infty).$$

Remark 3.2.16. *Generally, is not easy to compute the injective tensor norms explicitly. Concrete computations can be done only on some specific examples.*

Previously we have seen that the space of bilinear form on $X \times Y$ can be identified with the dual space of a projective tensor product of X and Y where (see for example [33])

$$\mathcal{B}(X \times Y) = \Big(X \otimes_\pi Y\Big)^*.$$

For finite-dimensional spaces, we can identify the injective and the projective tensor product by duality:

$$\Big(X \otimes_\pi Y\Big)^* = X^* \otimes_\varepsilon Y^*.$$

With this identification, we say that the injective and the projective tensor norms are dual.

3.2.3 General tensor norms

In the following section, we will give the general description of norms defined on the tensor product of two Banach spaces Banach spaces[1] $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. Such norms are known as *reasonable crossnorms* or simply *tensor norms*.[2]

Definition 3.2.17. *[33] Let* $(X, \|\cdot\|_X)$*,* $(Y, \|\cdot\|_Y)$ *Banach spaces and consider their dual spaces* $(X^*, \|\cdot\|_{X^*})$ *and* $(Y^*, \|\cdot\|_{Y^*})$*. We say that a norm* α *on* $X \otimes Y$ *is a reasonable crossnorm if it has the following properties:*

- *For* $u = x \otimes y \in X \otimes Y$ *we have:*

$$\|u\|_{X\otimes_\alpha Y} \leq \|x\|_X \, \|y\|_Y.$$

- *For every* $\varphi \in X^*$ *and* $\psi \in Y^*$*, the linear functional* $\varphi \otimes \psi$ *on* $X \otimes Y$ *is bounded and satisfies:*

$$\|\varphi \otimes \psi\|_{X^*\otimes_{\alpha^*} Y^*} \leq \|\varphi\|_{X^*} \, \|\psi\|_{Y^*}.$$

where α^* *is the dual norm of* α *defined on* $X^* \otimes Y^*$*, see equation (3.1).*

Remark 3.2.18. *The tensor norm* α *endows the space* $X \otimes Y$ *with a norm, hence it is a Banach space denoted by* $X \otimes_\alpha Y := (X \otimes Y, \|\cdot\|_{X\otimes_\alpha Y})$*. The dual tensor norm* α^* *is a tensor norm that endows the space* $X^* \otimes Y^*$ *with a norm, hence it also becomes a Banach space that we shall denote it by* $X^* \otimes_{\alpha^*} Y^* := (X^* \otimes Y^*, \|\cdot\|_{X^*\otimes_{\alpha^*} Y^*})$*. The two norms* α *and* α^* *define a duality between the Banach spaces* $X \otimes_\alpha Y$ *and* $X^* \otimes_{\alpha^*} Y^*$*:*

$$\Big(X \otimes_\alpha Y\Big)^* = X^* \otimes_{\alpha^*} Y^*.$$

Proposition 3.2.19. *[33] Let* X *and* Y *Banach spaces.*

[1] Or more than two.
[2] In this thesis the two terminologies are used.

- *A norm α on $X \otimes Y$ is a reasonable crossnorm if and only if:*

$$\|u\|_{X\otimes_\varepsilon Y} \le \|u\|_{X\otimes_\alpha Y} \le \|u\|_{X\otimes_\pi Y}.$$

  for every $u \in X \otimes Y$.

- *If α is a reasonable crossnorm on $X \otimes Y$ then $\|x \otimes y\|_{X\otimes_\alpha Y} = \|x\|_X \|y\|_Y$ for every $x \in X$ and $y \in Y$. Furthermore, for every $\varphi \in X^*$ and $\psi \in Y^*$, the linear functional $\varphi \otimes \psi$ on $X \otimes_\alpha Y$ satisfies:*

$$\|\varphi \otimes \psi\|_{X^*\otimes_{\alpha^*} Y^*} = \|\varphi\|_{X^*} \|\psi\|_{Y^*}.$$

Remark 3.2.20. *Previously, we have shown that the injective and the projective tensor norm satisfy the metric mapping property.*

3.2.4 Grothendieck constant

One of the most important results in the theory of tensor products of Banach spaces is due to Grothendieck [36–38]. In the following subsection, we shall introduce the Grothendieck constant that plays an essential role in different fields of physics, mathematics, and computer science; for more details on the topic see Pisier's extensive note on the subject [38]. Actually, there exist different versions of Grothendieck's theorem (see [38]), we will only give the statement that will be useful in its tensor norm formulation. Before we give Grothendieck's theorem, we shall define the following reasonable norm that will play an important role in the setting of nonlocal games in Chapter 4.

Definition 3.2.21. *Let two finite-dimensional Banach spaces X and Y with their respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, define the γ_2 tensor norm of $u \in X \otimes Y$ by:*

$$\|u\|_{X\otimes_{\gamma_2} Y} := \inf\left\{ \sup_{\alpha^*\in\mathbb{B}(X^*)} \left(\sum_{i=1}^N |\alpha^*(x_i)|^2\right)^{\frac{1}{2}} \sup_{\beta^*\in\mathbb{B}(Y^*)} \left(\sum_{j=1}^N |\beta^*(y_j)|^2\right)^{\frac{1}{2}} : u = \sum_{i=1}^N x_i \otimes y_i \right\}.$$

where the infimum is taken over all decompositions of $u = \sum_{i=1}^N x_i \otimes y_i$ with $x_i \in X$ and $y_j \in Y$. We write $X \otimes_{\gamma_2} Y = (X \otimes Y, \|\cdot\|_{X\otimes_{\gamma_2} Y})$, the Banach space induced by the γ_2 tensor norm on $X \otimes Y$.

As we have seen previously in this chapter, for a given tensor norm we have its natural dual tensor norm. In the following, we will give the definition of the dual tensor norm of γ_2.

Definition 3.2.22. *Let the Banach space $X \otimes_{\gamma_2} Y$ and $M \in X^* \otimes Y^*$, we define the dual γ_2^* norm of the γ_2 norm as:*

$$\|M\|_{X^*\otimes_{\gamma_2^*} Y^*} := \sup\Big\{ |\langle M, u\rangle| : \|u\|_{X\otimes_{\gamma_2} Y} \le 1 \Big\}.$$

We write the $X^* \otimes_{\gamma_2^*} Y^* := (X^* \otimes Y^*, \|\cdot\|_{X^*\otimes_{\gamma_2^*} Y^*})$, the Banach space induced by the γ_2^* tensor norm on $X^* \otimes Y^*$.

In what follows we give one version of the statements of Grothendieck's theorem where $X = Y = \ell_1^N(\mathbb{R})$.

Theorem 3.2.23. *[38](Grothendieck's theorem)*

Let $M \in \ell_1^N(\mathbb{R}) \otimes \ell_1^N(\mathbb{R})$, then there exists a positive universal constant $K_G^{\mathbb{R}}$ such that for every natural number N the following inequality holds:

$$\|M\|_{\ell_1^N(\mathbb{R})\otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})} \le K_G^{\mathbb{R}} \|M\|_{\ell_1^N(\mathbb{R})\otimes_\varepsilon \ell_1^N(\mathbb{R})}.$$

As we have mentioned previously that there exists a different version of the theorem, where here we have used the special Hilbert space as Banach space where the norm is induced by the scalar product, and the other versions are defined for general Banach spaces. Also, we shall mention that there exists a complex version of the theorem where the Grothendieck constant becomes $K_G^{\mathbb{C}}$. Actually, the value of $K_G^{\mathbb{R}}$ and $K_G^{\mathbb{C}}$ are not known exactly, computing their precise values as being still an open problem. However it is known that $K_G^{\mathbb{C}} < K_G^{\mathbb{R}}$, and that $K_G^{\mathbb{R}}$ verifies

$$1.67696\cdots \leq K_G^{\mathbb{R}} < \frac{\pi}{2\log(1+\sqrt{2})} = 1.7822139781\cdots .$$

For more details and discussion on the value of $K_G^{\mathbb{R}}$ and $K_G^{\mathbb{C}}$ see [38] and for a brief introduction of the different numerical values see [39].

Chapter 4

Nonlocality in quantum theory

The quantum revolution of the 20th century has not only allowed us to understand the microscopic world but also to turn upside down the concepts that were most deeply rooted in physics before quantum physics, such as locality. In classical physics, if two physical systems are separate and do not interact with each other through a force or a signal, the set of transformations or experiments on one will not affect the other, this is what we define by the *principle of locality*. We can ask ourselves the same question if we take two quantum systems: is the principle of locality respected? It turns out that quantum mechanics is intrinsically *nonlocal*, in the sense that any transformation and measurement on one of the two systems will affect the other. Even if one separates them far enough, the two systems remain intrinsically connected. One can think that in the microscopic world, the description of two systems must be considered as a total system, no matter the physical distance separating them. This concept was challenged by one of the fathers of quantum mechanics: Albert Einstein, with Podolsky and Rosen in [40]. The question of whether quantum mechanics is really intrinsically non-local has become more of a philosophical than a physical question. Then John Bell in his article [41] shows that any theory respecting the concept of locality must also respect a very precise statistical inequality. By using the principles of quantum mechanics, Bell's inequality can be violated. Thus Bell's result shows explicitly that based on the principle of locality and those of quantum mechanics quantum theory is intrinsically nonlocal, thus ending with a *mathematical criterion* the philosophical. This fact was also confirmed experimentally by Alain Aspect[1] in [42], and more recently in [43].

In Section 4.1 we shall introduce the historical derivation of a Bell inequality for the CHSH[2]experiments. In Section 4.2 we will give a general description of Bell inequality, we will introduce the different types of *correlations* used and some of their mathematical properties where we give the known geometrical interpretation of a Bell inequality. In Section 4.3, we introduce another point of view on Bell inequalities that can be understood as nonlocal games, where such games represent thought experiments, and we will show that we can recover the CHSH inequality and its violation in this framework. In Section 4.4, we will see that nonlocal games can be intrinsically related to the structure of tensor products of Banach spaces, where the Grothendieck constant plays a crucial role in separating the classical world description from the quantum one.

4.1 Historical derivation

In this section, we shall introduce the notions of *locality* and *nonlocality*. We will see that if a theory is described in a classical way, an inequality follows, while if one uses the intrinsic

[1]Thirty years later than Bells result, and allowed him recently to share the 2022 physics Nobel Prize with John Clauser and Anton Zeillinger.

[2]CHSH stands for John Clauser, Michael Horne, Abner Shimony, and Richard Holt.

quantum description of the world, the said inequality can be violated. In this section, we will only focus on the CHSH inequality which is a particular Bell inequality.

In order to describe Bell's inequality, we can imagine that for two sufficiently separate physical systems, two physicists Alice and Bob will make a certain number of measurements where they are not allowed to exchange any information. Thus the two physicists will start collecting a certain number of results a and b corresponding to x and y respectively. The whole system is described by a joint probability distribution between Alice and Bob, which corresponds to the probability of answering a and b for a certain pair of inputs x and y that we will note by: $\mathbb{P}(a\,b|x\,y)$.

If the systems are classical, *the locality* assumption assures the existence of some variables λ, having a joint causal influence on both outcomes, and which fully account for the dependence between the outcomes a and b. The variable λ actually characterizes all the uncontrolled physical degrees of freedom, where it is natural to think that λ is sampled with a probability measure μ. Due to locality constraints, the joint probability distribution $\mathbb{P}(a\,b|x\,y)$ factorizes as follows:

$$\mathbb{P}_l(a\,b|x\,y) = \int_\Lambda d\mu(\lambda)\,\mathbb{P}_{\mathbb{A}}(a|x\,,\lambda)\,\mathbb{P}_{\mathbb{B}}(b|y\,,\lambda).$$

with $\mathbb{P}_l(a\,b|x\,y)$ stands for the local joint probability distribution. In the equation above, $\mathbb{P}_{\mathbb{A}}(a|x\,,\lambda)$ is Alice's probability of observing the outcome a for a given measurement x and the hidden random variable λ, and the same for Bob with $\mathbb{P}_{\mathbb{B}}(b|y\,,\lambda)$.

For simplicity, we can consider an experiment where there are only two measurement choices by Alice and Bob $x, y \in \{0,1\}$ with only two possible outcomes a and b taking also two values $a, b \in \{-1,1\}$.

Let consider the following quantity:

$$S := \langle a_0\,b_0\rangle + \langle a_0 b_1\rangle + \langle a_1 b_0\rangle - \langle a_1 b_1\rangle$$

with $\langle a_x b_y\rangle := \sum_{a,b} a\,b\,\mathbb{P}(a\,b|x\,y)$.

If Alice and Bob use the locality assumption of their theory one can show that the following inequality holds:

$$S = \langle a_0 b_0\rangle + \langle a_0 b_1\rangle + \langle a_1 b_0\rangle - \langle a_1 b_1\rangle \leq 2.$$

The inequality above is known as the *CHSH inequality*, are its physical meaning is very profound. It says that for any physical theory with the locality assumption, the quantity S_l is always bounded by 2. It is well known since John Bells' work [41] that the quantum theory violates this bound.

Let us assume that the external physical reality is governed by the principle of quantum theory, and let us say that the two systems form an entangled pair

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle).$$

Alice performing a spin measurement $\sigma = (\sigma_1, \sigma_2, \sigma_3)$[3]on the direction $\mathbf{x}$, hence she will perform the measurement $\mathbf{x}\cdot\sigma$. Bob will also perform a spin measurement over another axis hence he will perform $\mathbf{y}\cdot\sigma$. Using the formalism of the quantum theory the probability of obtaining the result a and b for a given measurement $x, y \in \{0,1\}$, Alice and Bob joint probability is given by

$$\mathbb{P}_Q(a\,b|x\,y) = \langle\psi|\,A_{a|x}\otimes B_{b|y}\,|\psi\rangle\,.$$

[3]We recall the Pauli matrices $\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

where $A_{a|x}$ and $B_{b|y}$ represent respectively Alice and Bob POVMs for a spin measurement in a given direction $\mathbf{x}$ and $\mathbf{y}$ given by:

$$A_{0|x} := \frac{1}{2}(I + \mathbf{x}\cdot\sigma)$$
$$A_{1|x} := \frac{1}{2}(I - \mathbf{x}\cdot\sigma)$$
$$B_{0|y} := \frac{1}{2}(I + \mathbf{y}\cdot\sigma)$$
$$B_{1|y} := \frac{1}{2}(I - \mathbf{y}\cdot\sigma)$$

For a choice of $x = 0$, Alice should perform the measurement on the direction $\mathbf{x} = \mathbf{e_1}$, while if $x = 1$ she should perform the measurement on $\mathbf{e_2}$ direction. Also for Bob for the choice of measurement on $y = 0$ he should do the measurement on $\mathbf{y} = \frac{1}{\sqrt{2}}(\mathbf{e_2} - \mathbf{e_1})$ and for $y = 1$ he should measure on $\mathbf{y} = \frac{-1}{\sqrt{2}}(\mathbf{e_2} + \mathbf{e_1})$.

One can check with the quantum probability distribution $\mathbb{P}_Q(ab|xy)$ we obtain the following result

$$\langle a_x b_y\rangle_Q = \sum_{a,b} a\, b\, \mathbb{P}_Q(ab|xy) = -\mathbf{x}\cdot\mathbf{y}.$$

We can construct the same quantity analogous to S_l which is given by

$$S_Q = \langle a_0 b_0\rangle_Q + \langle a_0 b_1\rangle_Q + \langle a_1 b_0\rangle_Q - \langle a_1 b_1\rangle_Q = 2\sqrt{2}.$$ [4]

we remark that

$$S_Q > S_l,$$

hence we say we have a *violation of the CHSH inequality.*

This result illustrates perfectly the difference between local theory and quantum theory, where it shows that one cannot find any local description of the quantum world.

4.2 Bell inequalities

As we have seen in the previous section, we can give a precise way to distinguish between classical and quantum theory through the CHSH inequality. It turns out that the CHSH inequality is a particular expression that mixes the expectation value of Alice and Bob measuring a and b for given inputs x and y in the classical theory while its violation detects quantum effects. In general, we study Bell inequalities (the CHSH one being a special case) that describe the separation between the classical and the quantum world. In the following section, we shall distinguish different types of *strategies*[5] which are the probability distributions used by Alice and Bob. We will introduce the *classical set*, *quantum set*, and the *non-signaling set* that goes beyond the quantum mechanical description of nature. These represent respectively the classical, quantum, and non-signaling probabilities that Alice and Bob will share as a strategy. We will briefly introduce the mathematical meaning of Bell inequalities.

When Alice and Bob run their experiments[6], one can imagine that they have some inputs denoted by $x, y \in \{1, \cdots, N\}$ and some outputs $a, b \in \{1, \cdots, M\}$[7]. The output represents the outcome when measuring the physical system, in the experiments the measurement apparatus

[4] Actually the protocol described to obtain the value $2\sqrt{2}$ is the optimal one.

[5] This terminology will make complete sense when we will introduce the framework of nonlocal games in Section 4.3.

[6] They run their experiments but when they collect data, they are space-like separated and not allowed to communicate.

[7] In general we can assume that Alice and Bob don't have the same number of outputs.

can be seen as a black box[8] that takes some inputs and gives the outputs. Such a thought experiment is known as a *Bell scenario*.

Definition 4.2.1. *We define the classical set $\mathcal{L}_{N,M}$, the set of all classical probabilities given by*

$$\mathcal{L}_{N,M} := \Big\{\mathbb{P}_l(a\,b|x\,y) \,\Big|\, \mathbb{P}_l(a\,b|x\,y) = \int_\Lambda d\mu(\lambda)\, \mathbb{P}_{\mathbb{A}}(a|x\,,\lambda)\, \mathbb{P}_{\mathbb{B}}(b|y\,,\lambda)\Big\}.$$

The classical set $\mathcal{L}_{N,M}$ *represents all the shared classical joint probabilities used by Alice and Bob during the experiments.*

If Alice and Bob perform experiments where quantum effects are not negligible, their joint probability distribution is derived from the quantum formalism. They will share a quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ with d_A and d_B being the dimensions of Alice and Bob's respective Hilbert space.

Their joint probability is given in general by

$$\mathbb{P}_Q(a\,b|x\,y) = \langle\psi|\, A_{a|x} \otimes B_{b|y}\, |\psi\rangle\,.$$

where $A_{a|x}$ denotes Alice's POVMs: for a given measurement x, she got the outcome a satisfying the normalisation condition $\sum_a A_{a|x} = I_{d_A}$ for all $x \in \{1, \cdots, M\}$. The same is for Bob's POVM $B_{b|y}$ where for a given measurement y he obtains the outcome b with the normalization $\sum_b B_{b|y} = I_{d_B}$.

Definition 4.2.2. *We define the quantum set $\mathcal{Q}_{N,M}$, the set of all quantum probabilities, by*

$$\mathcal{Q}_{N,M} := \Big\{\mathbb{P}_Q(a\,b|x\,y) \,\Big|\, \mathbb{P}_Q(a\,b|x\,y) = \langle\psi|A_{a|x} \otimes B_{b|y}\,|\psi\rangle\,;\, A_{a|x}, B_{b|y} \geq 0, \\ \forall x \sum_{a=1}^{M} A_{a|x} = I_{d_A}; \forall y \sum_{b=1}^{M} B_{b|y} = I_{d_B}\Big\}.$$

The quantum set $\mathcal{Q}_{N,M}$ *represents all the shared joint quantum probabilities used by Alice and Bob during the experiments.*

One can imagine also that Alice and Bob can use correlations that go beyond the quantum setting. Since they are space-like and separated, they cannot communicate by exchanging signals (this is not allowed by the principles of special relativity). The non signaling condition is given by

$$\sum_b \mathbb{P}_{NS}(a\,b|x\,y) = \sum_b \mathbb{P}_{NS}(a\,b|x\,y') \quad \forall a, x, y', y.$$

and

$$\sum_a \mathbb{P}_{NS}(a\,b|x\,y) = \sum_a \mathbb{P}_{NS}(a\,b|x'\,y) \quad \forall b, x, x', y.$$

The non-signaling condition implies that Alice and Bob's probability distribution are independent in the following sense

$$\mathbb{P}(a|x) = \sum_b \mathbb{P}_{NS}(a\,b|x\,y)$$

does not depend on the question y that Bob received. Bob's probability distribution is given by

$$\mathbb{P}(b|y) = \sum_a \mathbb{P}_{NS}(a\,b|x\,y),$$

and similarly, does not depend on the question x that Alice received. The non-signaling condition gives rise to correlations that go beyond the quantum setting.

[8]The black box could be understood as an operation where we don't know the microscopic details of the measurement apparatus.

Definition 4.2.3. *The non signaling set $\mathcal{NS}_{N,M}$, the set of all non signaling probabilities, is given by*

$$\mathcal{NS}_{N,M} = \Big\{ \mathbb{P}_{NS}(a\,b|x\,y) \Big| \sum_a \mathbb{P}_{NS}(a\,b|x\,y) = \sum_a \mathbb{P}_{NS}(a\,b|x'\,y)\,\forall b, x, x', y$$
$$\text{and} \sum_b \mathbb{P}_{NS}(a\,b|x\,y) = \sum_b \mathbb{P}_{NS}(a\,b|x\,y')\,\forall a, x, y', y \Big\}.$$

The non-signaling set $\mathcal{NS}_{N,M}$ represents all the shared joint non-signaling probabilities used by Alice and Bob during the experiments.

We shall just mention a few properties of the space of correlations that we have considered above and the mathematical description of a Bell inequality. Actually one can show that $\mathcal{L}_{N,M} \subset \mathcal{Q}_{N,M} \subset \mathcal{NS}_{N,M}$. Let $\mathcal{K}$ a correlation set that can be either $\mathcal{L}_{N,M}$,$\mathcal{Q}_{N,M}$ or $\mathcal{NS}_{N,M}$. It can be seen easily that all the spaces of correlation are *convex*, $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{K}$ then $\mu\mathbb{P}_1 + (1-\mu)\mathbb{P}_2 \in \mathcal{K}$ for $\mu \in [0,1]$. Actually, it was shown in [44] that $\mathcal{L}_{N,M}$ and $\mathcal{NS}_{N,M}$ are *polytopes*, defined by a convex combination of finitely many extreme points. Bell inequalities are hyperplanes separating the classical set $\mathcal{L}_{N,M}$. Actually searching for all Bell inequalities geometrically can be understood as studying all the possible facets of the polytope $\mathcal{L}_{N,M}$. It turns out that such a problem is a difficult task we shall just refer to the review paper [45] and the references therein for more details.

4.3 Nonlocal games

We recall from Section 4.1, that when one uses the classical correlation set we obtain the famous CHSH inequality, given by:

$$S_l = \langle a_0 b_0 \rangle_l + \langle a_0 b_1 \rangle_l + \langle a_1 b_0 \rangle_l - \langle a_1 b_1 \rangle_l \leq 2.$$

If one uses the quantum correlation set one obtains a violation of the inequality above

$$S_Q = \langle a_0 b_0 \rangle_Q + \langle a_0 b_1 \rangle_Q + \langle a_1 b_0 \rangle_Q - \langle a_1 b_1 \rangle_Q = 2\sqrt{2} > S_l.$$

Actually, there exists another way of understanding Bell inequalities by using the framework of *nonlocal games*.

4.3.1 General description

In this subsection, we shall present the *nonlocal game* framework. It will play an important role in this manuscript since it gives a natural way of understanding Bell inequalities. We will also show in the following section that the framework of nonlocal games is intrinsically related to the structure of tensor products of Banach spaces.

As in any game, we need players and we need *rules*. In the nonlocal game framework, the players are Alice and Bob[9], and the *referee* dictates the rules of the game, which correspond to a *payoff*. Before the game starts, Alice and Bob are allowed to choose a *strategy* for playing the game, where the strategy consists of choosing one of the different correlation sets: either classical, quantum, or non-signaling that will correspond respectively to the classical quantum on non-signaling strategies. When they choose their strategies, they separate [10] and are not allowed to communicate anymore. When the game starts, the referee asks a pair of questions to

[9] Actually we can add more players but for simplicity, we will restrict to only two players. However, in [46] they have shown if we add only another player to the story, the maximal of the game with the quantum setting diverges with the number of questions (see Theorem 4.4.12).

[10] They are space-like separated.

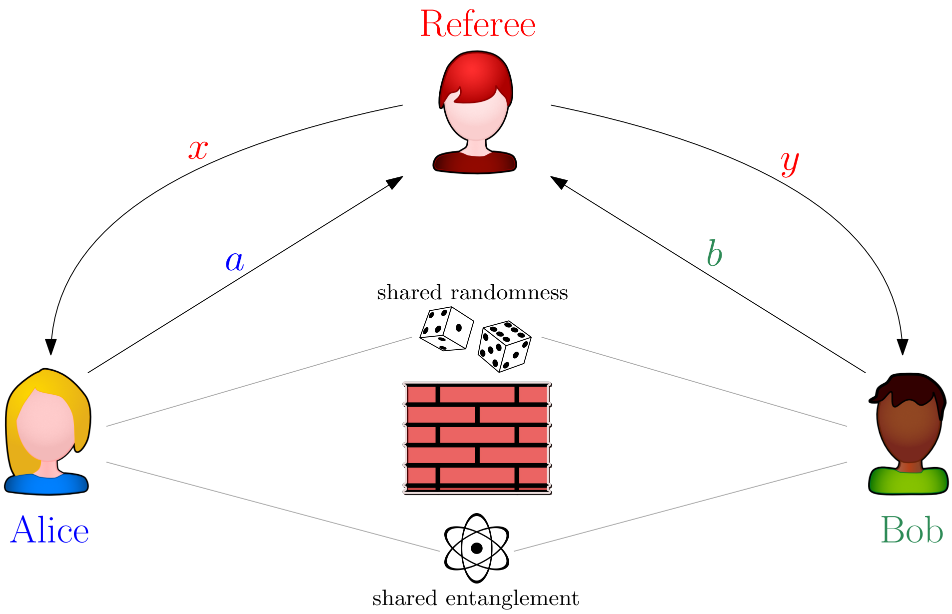


Figure 4.1: Nonlocal game where Alice and Bob use classical (shared randomness) or quantum (shared entanglement) strategies.

Alice and Bob. We shall denote the set of questions for Alice by $\mathcal{X}$ and for Bob by $\mathcal{Y}$. The referee chooses a pair of questions randomly with a probability distribution $\pi : \mathcal{X} \times \mathcal{Y} \to \pi(x, y) \in [0, 1]$ where $x \in \mathcal{X}$ denotes the question that the referee sends to Alice and $y \in \mathcal{Y}$ to Bob. When Alice and Bob receive their questions they generate some answers or outcomes $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The referee will receive Alice's and Bob's answers and he will decide if they are correct or wrong; this corresponds to the payoff. The payoff is given by $V : \mathcal{X} \times \mathcal{Y} \times \mathcal{A} \times \mathcal{B} \to V(x, y, a, b)$ where $V(x, y, a, b)$ can be either 0 or 1 depending on whether the players lose or win the game; for an illustration of a nonlocal game, see the Figure 4.1.

To summarise, a game $\mathcal{G}$ is completely characterized by $\mathcal{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \pi, V)$, with two players Alice and Bob space like separated and a referee. As we have seen before, the players Alice and Bob can either choose classical, quantum, or non-signaling strategies.

Alice's and Bob's expected payoff is given by

$$\begin{aligned} \omega(G, \mathbb{P}) &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} \pi(x, y)\, V(x, y, a, b)\, \mathbb{P}(a\,b|x\,y) \\ &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} G^{a,b}_{x,y}\, \mathbb{P}(a\,b|x\,y). \end{aligned}$$

with $G^{a,b}_{x,y} = \pi(x, y)\, V(x, y, a, b)$.

Definition 4.3.1. *In this definition, we give the classical, quantum, and non-signaling values of the game.*

- *The* classical value of the game $\omega(G)$ *is defined as the maximal payoff when the players use the classical set, where Alice and Bob optimize over all possible classical probabilities:*

$$\omega(G) = \sup_{\mathbb{P}_l(a\,b|x\,y)\in\mathcal{L}_{N,M}} |\omega(G, \mathbb{P}_l)|.$$

- *The* quantum value of the game $\omega^*(G)$ *is defined as the maximal payoff when the players use the quantum set, where Alice and Bob optimize over all possible quantum probabilities:*

$$\omega^*(G) := \sup_{\mathbb{P}_Q(a\,b|x\,y)\in\mathcal{Q}_{N,M}} |\omega(G, \mathbb{P}_Q)|.$$

- *The* non-signaling value *of the game $\omega_{\mathcal{NS}}(G)$ is defined as the maximal payoff when the players use the non-signaling set, where Alice and Bob optimize over all possible non-signaling probabilities:*

$$\omega_{\mathcal{NS}}(G) := \sup_{\mathbb{P}_{NS}(a\,b|x\,y)\in\mathcal{NS}_{N,M}} |\omega(G, \mathbb{P}_{NS})|.$$

Remark 4.3.2. *In the rest of the manuscript, we will only focus on the classical and quantum setting.*

4.3.2 CHSH as a nonlocal game

The CHSH game can also be understood as a nonlocal game. There exist different ways of obtaining the different classical and quantum values of the CHSH game. Here we will give the first derivation of (in)compatibility[11] and the CHSH inequality with the game description [47]. In this setting, as before Alice and Bob are space-like separated and not allowed to communicate. The referee will ask respectively to Alice and Bob a tuple of questions from the question set $\mathcal{X}$ and $\mathcal{Y}$. And the players will give their respective answers from $\mathcal{A}$ and $\mathcal{B}$. The CHSH game is given by $\mathcal{X} = \mathcal{Y} = \mathcal{A} = \mathcal{B} = \{\pm 1\}$, or two input two output games. The referee will choose the questions uniformly, with $\pi(x, y) = \frac{1}{4}$, for all x, y. The payoff is given by

$$V(x, y, a, b) = \begin{cases} 1 & a \oplus b = x \cdot y \\ 0 & \text{otherwise} \end{cases}$$

When calculating $\omega(G_{CHSH}, \mathbb{P}_Q)$, one can show easily that we obtain the following quantity

$$\begin{aligned} \omega(G_{CHSH}, \mathbb{P}_Q) &= \frac{1}{4} \sum_{a,b} \sum_{x,y} V(x, y, a, b)\ \langle\psi|\, A_{a|x} \otimes B_{b|y} \,|\psi\rangle \\ &= \langle\psi|\, A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 \,|\psi\rangle\,. \end{aligned}$$

with $A_i := A_{1|i} - A_{-1|i}$ and $B_i := B_{1|i} - B_{-1|i}$ are respectively Alice's and Bob's measurement observables[12]. Define the operator C as

$$C := A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1.$$

One can easily compute C^2:

$$C^2 = 4I_{AB} - [A_0, A_1] \otimes [B_0, B_1].$$

with $I_{AB} = I_A \otimes I_B$.

The optimization over the shared quantum correlation, is obtained by optimizing over all the shared quantum states and measurement apparatuses, hence we shall be interested in computing $\|C^2\|_\infty$.

Remark that if $[A_0, A_1] = 0$ or $[B_0, B_1] = 0$, which is equivalent to *the classical setting*, we obtain the classical inequality:

$$\|C^2\|_\infty = \|C\|_\infty^2 \leq 4. \iff \|C\|_\infty \leq 2.$$

Hence the *classical value of the game*, corresponding to the maximal we can obtain using the classical correlations, is

$$\omega(G_{CHSH}) \leq 2.$$

[11]Compatibility here, means non-commutativity, but it is well known that they are not the same see Chapter 5.

[12]With $(A_{1|i}, A_{-1|i})$ is Alice's PVM satisfying $A_{1|i} + A_{-1|i} = I_d$ and the same for Bob.

Now in the quantum setting, we obtain the following bound

$$\begin{aligned}\|C^2\|_\infty &= \|A_0\otimes B_0 + A_0\otimes B_1 + A_1\otimes B_0 - A_1\otimes B_1\| \\ &\leq 4 + 4\|A_0\|\|A_1\|\|B_0\|\|B_1\| = 8.\end{aligned}$$

Using again as before $\|C^2\|_\infty = \|C\|_\infty^2$ we have

$$\|C\|_\infty \leq 2\sqrt{2}$$

where the maximal quantum value for the CHSH game is

$$\omega^*(G_{CHSH}) \leq 2\sqrt{2}.$$

Tsirelson showed in [48] that the optimal violation of the CHSH game is obtained by performing a measurement on Pauli matrices and uses a maximally entangled state. In this way we recover the famous Bell inequality violation:

$$\omega^*(G_{CHSH}) = 2\sqrt{2} > \omega(G_{CHSH}) = 2.$$

4.4 Nonlocal games and Tensor norms

In this section, we shall describe the intrinsic mathematical framework of Bell inequalities. Bell inequalities are deeply related to the theory of tensor norms. In this subsection, we will give an overview of such a description. In the framework of nonlocal games, they can be described naturally in such a framework, where we will see also that the Grothendieck constant plays an important role.

We recall from the Section 4.3 that a game $\mathcal{G}$ is given by $\mathcal{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \pi, V)$, with two players Alice and Bob space like separated and a referee. The referee will ask a pair of questions $x \in \mathcal{X}$ to Alice and $y \in \mathcal{Y}$ to Bob. In this section, we will only be allowed the players Alice and Bob to choose either classical or quantum strategies, and their maximal classical and the quantum expected payoff is given respectively by $\omega(G)$ and $\omega^*(G)$ (see Definition 4.3.1).

4.4.1 XOR games

In the following subsection, we shall investigate an important class of games known as *XOR games*. XOR games or two-player XOR games are a class of games where Alice's and Bob's answers are binary in the sense that $\mathcal{A} = \mathcal{B} = \{0, 1\}$ and the payoff $V(x, y, a, b) := \frac{1}{2}(1 + (-1)^{a\oplus b\oplus c_{x,y}})$ with $c_{x,y} \in \{0, 1\}$.

We shall compute the payoff of Alice and Bob in the XOR setting:

$$\begin{aligned}\omega(G,\mathbb{P}) &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} \pi(x,y)\, V(x,y,a,b)\, \mathbb{P}(a\, b|x\, y) \\ &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \sum_{(a,b)\in\{0,1\}^2} \pi(x,y)\frac{1}{2}(1 + (-1)^{a\oplus b\oplus c_{xy}})\mathbb{P}(a\, b|x\, y) \\ &= \frac{1}{2} + \frac{1}{2}\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \pi(x,y)(-1)^{c_{x,y}}(\mathbb{P}(0\,0|x\,y) + \mathbb{P}(1\,1|x\,y) - \mathbb{P}(0\,1|x\,y) - \mathbb{P}(1\,0|x\,y)) \\ &= \frac{1}{2} + \frac{\beta(G,\mathbb{P})}{2},\end{aligned}$$

where we have defined the *bias* of the XOR game as

$$\beta(G,\mathbb{P}) := \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y}\,(\mathbb{P}(0\,0|x\,y) + \mathbb{P}(1\,1|x\,y) - \mathbb{P}(0\,1|x\,y) - \mathbb{P}(1\,0|x\,y)) \in [-1, 1]$$

and $G_{x,y} := \pi(x,y)(-1)^{c_{x,y}}$.

Generally in the setting of XOR games we are interested in computing the bias of the game instead of the payoff; these quantities are related by $\beta(G,\mathbb{P}) = 2\omega(G,\mathbb{P}) - 1$.

As for the classical and quantum value of the game, we shall introduce the *classical bias* and the *quantum bias*[13].

Definition 4.4.1. *The* classical bias $\beta(G)$ *of the XOR game is given as the maximal value of the bias* $\beta(G,\mathbb{P})$ *over the classical shared probabilities:*

$$\beta(G) := \sup_{\mathbb{P}_l(a\,b|x\,y)\in\mathcal{L}} |\beta(G,\mathbb{P}_l)|.$$

Definition 4.4.2. *The* quantum bias $\beta^*(G)$ *of the XOR game is given as the maximal value of the bias* $\beta(G,\mathbb{P})$ *over the quantum shared probabilities:*

$$\beta^*(G) := \sup_{\mathbb{P}_Q(a\,b|x\,y)\in\mathcal{Q}} |\beta(G,\mathbb{P}_Q)|.$$

In the following, we shall compute the classical and quantum bias, we will show that optimization over the classical set and the quantum set is the same as optimizing on the *classical and the quantum correlation* that we will introduce below.

The *classical bias* $\beta(G)$ of the game is given by:

$$\begin{aligned}
\beta(G) &:= \sup_{\mathbb{P}_l(a\,b|x\,y)\in\mathcal{L}} |\beta(G,\mathbb{P}_L)| \\
&= \sup_{\mathbb{P}_l(a\,b|x\,y)\in\mathcal{L}} \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y}(\mathbb{P}(0\,0|x\,y) + \mathbb{P}(1\,1|x\,y) - \mathbb{P}(0\,1|x\,y) - \mathbb{P}(1\,0|x\,y)) \Big| \\
&= \sup \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y} \int_\Lambda d\mu(\lambda) \sum_{(a,b)\in\mathbb{Z}_2\times\mathbb{Z}_2} ab\,\mathbb{P}_{\mathbb{A}}(a|x,\lambda)\,\mathbb{P}_{\mathbb{B}}(b|y,\lambda) \Big| \\
&= \sup \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y} \int_\Lambda d\mu(\lambda) A_x(\lambda)\, B_y(\lambda) \Big| \\
&= \sup_{\gamma_{x,y}} \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y}\,\gamma_{x,y} \Big|.
\end{aligned}$$

where we have used $A_x(\lambda) = \sum_{a\in\{0,1\}} a\,\mathbb{P}_{\mathbb{A}}(a|x,\lambda)$, $B_y(\lambda) = \sum_{b\in\{0,1\}} b\,\mathbb{P}_{\mathbb{B}}(b|y,\lambda)$, and $\gamma_{x,y} := \int_\Lambda d\mu(\lambda) A_x(\lambda)\, B_y(\lambda)$.

Remark that $|A_x(\lambda)|, |B_y(\lambda)| \leq 1$, and the last supremum is on $\gamma_{x,y}$, which motivates the following definition of *classical correlation set*.

Definition 4.4.3. *We define the classical correlation set as*

$$\mathbb{L}_N := \left\{ \gamma_{x,y} \Big| \gamma_{x,y} = \int_\Lambda A_x(\lambda) B_y(\lambda)\, \mathrm{d}\mu(\lambda);\ |A_x(\lambda)|, |B_y(\lambda)| \leq 1 \right\} \subseteq \mathcal{M}_N(\mathbb{R}).$$

With the definition of *classical correlation set* above, we can give an equivalent definition of the classical bias given in the Definition 4.4.4.

Definition 4.4.4. *The* classical bias $\beta(G)$ *of the XOR game is defined as:*

$$\beta(G) := \sup_{\gamma_{x,y}\in\mathbb{L}_N} \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y}\,\gamma_{x,y} \Big|.$$

[13] We can also consider the non-signaling bias "equivalent" to the non-signaling value of the game.

Similarly to the classical bias $\beta(G)$, we shall compute the quantum bias $\beta^*(G)$ for XOR games:

$$\begin{aligned}\beta^*(G) &:= \sup_{\mathbb{P}_Q(a\,b|x\,y)\in\mathcal{Q}} |\beta(G,\mathbb{P}_Q)| \\ &= \sup_{\mathbb{P}_Q(a\,b|x\,y)\in\mathcal{Q}} \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y}(\mathbb{P}(0\,0|x\,y)+\mathbb{P}(1\,1|x\,y)-\mathbb{P}(0\,1|x\,y)-\mathbb{P}(1\,0|x\,y))\Big|.\end{aligned}$$

One can check easily that:

$$\begin{aligned}\mathbb{P}(0\,0|x\,y)+\mathbb{P}(1\,1|x\,y)-\mathbb{P}(0\,1|x\,y)-\mathbb{P}(1\,0|x\,y) &= \langle\psi|\, A_{0|x}\otimes B_{0|y}\,|\psi\rangle + \langle\psi|\, A_{1|x}\otimes B_{1|y}\,|\psi\rangle \\ &\quad - \langle\psi|\, A_{0|x}\otimes B_{1|y}\,|\psi\rangle - \langle\psi|\, A_{1|x}\otimes B_{0|y}\,|\psi\rangle \\ &= \langle\psi|\, A_x\otimes B_y\,|\psi\rangle\end{aligned}$$

where we have defined Alice's and Bob's respective *measurement observables* by $A_x := A_{0|x} - A_{1|x} \in [-I, I]$ and $B_y := B_{0|y} - B_{1|y} \in [-I, I]$.

Hence the quantum bias is given by

$$\begin{aligned}\beta^*(G) &= \sup \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y}\,\langle\psi|\, A_x\otimes B_y\,|\psi\rangle\,\Big| \\ &= \sup_{\gamma_{x,y}} \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y}\,\gamma_{x,y}\Big|,\end{aligned}$$

where the last supremum is taken on $\gamma_{x,y}$.

The computation above motivates the following definition of the *quantum correlation set.*

Definition 4.4.5. *We define the quantum correlation set as*

$$\mathbb{Q}_N := \Big\{\gamma_{x,y}\Big|\gamma_{x,y} = \langle\psi|\, A_x\otimes B_y\,|\psi\rangle\,;\, \|\psi\|=1,\, \|A_x\|_\infty, \|B_y\|_\infty \le 1\Big\} \subseteq \mathcal{M}_N(\mathbb{R}).$$

The definition of the *quantum correlation set* allows giving an equivalent formulation of the Definition 4.4.2.

Definition 4.4.6. *The quantum bias $\beta^*(G)$ of an XOR game is defined as:*

$$\beta^*(G) = \sup_{\gamma_{x,y}\in\mathbb{Q}_N} \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y}\,\gamma_{x,y}\Big|.$$

The classical bias and the quantum bias $\beta(G)$ and $\beta^*(G)$ describe respectively the maximal value of the game by using classical or quantum correlations.

4.4.2 Nonlocality and Banach space theory

In this subsection, we will show the link between the bias of XOR game and the tensor product of Banach spaces. We will show that depending on the classical or the quantum strategies one can associate it with a specific norm on the tensor of the game G.

For the reader's convenience, we shall recall the important concept that we have introduced in Chapter 3. Let $u \in X\otimes Y$, and α tensor norm on $X\otimes Y$ (see the definition 3.2.17 and the proposition 3.2.19), we note the norm α of u as $\|u\|_{X\otimes_\alpha Y}$. We recall, the dual tensor norm α^* of a given tensor norm α on $X\otimes Y$ is given by $u \to \|u\|_{X\otimes_{\alpha^*} Y} := \sup\{|\langle v, u\rangle| \,:\, \|v\|_{X^*\otimes_\alpha Y^*} \le 1\}$.

From the previous subsection, we have shown that the classical bias $\beta(G)$ is given by

$$\beta(G) = \sup_{\gamma_{x,y}\in\mathbb{L}_N} \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y}\,\gamma_{x,y}\Big|.$$

To put the link with the Banach space theory, we can rewrite the classical bias as

$$\beta(G) := \sup_{\gamma\in\mathbb{L}_N} |\langle G, \gamma\rangle|$$

Remark 4.4.7. *Remark that the classical correlation set* $\mathbb{L}_N$ *is the convex hull of* $(a_x\, b_y)_{x,y=1}^N$*:*

$$\mathbb{L}_N = \operatorname{Conv}\left\{ (a_x b_y)_{x,y=1}^N \middle| a_x = b_y = \pm 1 \right\}$$

Theorem 4.4.8. *The classical bias is completely characterized by the injective norm of the game* G *and we have:*

$$\beta(G) = \|G\|_{\ell_1^N(\mathbb{R})\otimes_\varepsilon \ell_1^N(\mathbb{R})}$$

and, by duality, the classical correlation set is

$$\mathbb{L}_N = \mathbb{B}\Big(\ell_\infty^N(\mathbb{R}) \otimes_\pi \ell_\infty^N(\mathbb{R})\Big).$$

where $\mathbb{B}(X)$ *is the unit ball of the Banach space* X*.*

Proof. Start from the classical bias of a given game G:

$$\begin{aligned}
\beta(G) &= \sup_{\gamma\in\mathbb{L}_N} |\langle G,\gamma\rangle| = \sup_{\gamma_{x,y}\in\mathbb{L}} \left| \sum_{x,y} G_{x,y}\gamma_{x,y} \right| \\
&= \sup_{a,b\in\mathcal{B}(\ell_\infty^N(\mathbb{R}))} \left| \sum_{x,y} G_{x,y}\langle e_x\otimes e_y, a\otimes b\rangle \right| \\
&= \sup_{a,b\in\mathcal{B}(\ell_\infty^N(\mathbb{R}))} \Big| \langle G, a\otimes b\rangle \Big| = \|G\|_{\ell_1^N(\mathbb{R})\otimes_\varepsilon \ell_1^N(\mathbb{R})}.
\end{aligned}$$

By using the duality we have

$$\|G\|_{\ell_1^N(\mathbb{R})\otimes_\varepsilon \ell_1^N(\mathbb{R})} = \sup\{\langle G,\gamma\rangle;\ \|\gamma\|_{\ell_\infty^N(\mathbb{R})\otimes_\pi \ell_\infty^N(\mathbb{R})} \le 1\},$$

where we recall that

$$\Big(\ell_1^N(\mathbb{R}) \otimes_\varepsilon \ell_1^N(\mathbb{R})\Big)^* = \ell_\infty^N(\mathbb{R}) \otimes_\pi \ell_\infty^N(\mathbb{R}),$$

hence trivially we have

$$\gamma \in \mathbb{B}\Big(\ell_\infty^N(\mathbb{R}) \otimes_\pi \ell_\infty^N(\mathbb{R})\Big) = \mathbb{L}_N,$$

which ends the proof of the theorem. □

The theorem above shows that all the information of the classical bias is completely described by the injective norm of two copies of $\ell_1^N(\mathbb{R})$.

To characterize in a similar way the quantum bias $\beta^*(G)$, we shall recall the tensor norm γ_2 from Definition 3.2.21.

Definition 4.4.9. *Let two finite-dimensional Banach spaces* X *and* Y *with their respective norms* $\|\cdot\|_X$ *and* $\|\cdot\|_Y$*, define the tensor norm* γ_2 *of* $u\in X\otimes Y$ *by:*

$$\|u\|_{X\otimes_{\gamma_2}Y} := \inf\left\{ \sup_{\alpha^*\in\mathbb{B}(X^*)} \left(\sum_{i=1}^N |\alpha^*(x_i)|^2\right)^{\frac12} \sup_{\beta^*\in\mathbb{B}(Y^*)} \left(\sum_{j=1}^N |\beta^*(y_j)|^2\right)^{\frac12} : u = \sum_{i=1}^N x_i\otimes y_i \right\}.$$

where the infimum is taken over all decompositions of $u = \sum_{i=1}^N x_i\otimes y_i$ *with* $x_i\in X$ *and* $y_j\in Y$*. We write* $X\otimes_{\gamma_2} Y = (X\otimes Y, \|\cdot\|_{X\otimes_{\gamma_2}Y})$ *for the Banach space induced by the* γ_2 *tensor norm on* $X\otimes Y$*.*

Theorem 4.4.10. *Let* z *be an* $N\times N$ *real matrix. The following statements are equivalent:*

- *z is in the quantum correlation set:* $z\in\mathbb{Q}_N$*.*

- *There exists norm one vectors $\{u_x\}_{x=1}^N$ and $\{v_y\}_{y=1}^N$ in a real Hilbert space H, such that:*

$$z_{x,y} = \langle u_x, v_y \rangle.$$

In particular we have

$$\beta^*(G) = \sup \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y} \langle u_x, v_y \rangle \Big|,$$

where the supremum is taken over the vectors u_x and v_y in the unit sphere of the real Hilbert space H.

With Theorem 4.4.10, Tsirelson has shown that the quantum bias can be written in a tensor norm formulation.

Theorem 4.4.11. *The quantum bias of the game G is completely described by the following norm:*

$$\beta^*(G) = \sup_{\gamma\in\mathbb{Q}} \{|\langle G, \gamma\rangle|\} = \|G\|_{\ell_1^N(\mathbb{R})\otimes_{\gamma_2^*}\ell_1^N(\mathbb{R})}$$

and by duality, the quantum correlation set is

$$\mathbb{Q}_N = \mathbb{B}\Big(\ell_\infty^N(\mathbb{R}) \otimes_{\gamma_2} \ell_\infty^N(\mathbb{R})\Big),$$

where we recall from Definition 3.2.22 the dual norm $\|G\|_{\ell_1^N(\mathbb{R})\otimes_{\gamma_2^*}\ell_1^N(\mathbb{R})}$ is given by:

$$\|G\|_{\ell_1^N(\mathbb{R})\otimes_{\gamma_2^*}\ell_1^N(\mathbb{R})} := \sup\Big\{|\langle G, \gamma\rangle| : \|\gamma\|_{\ell_\infty^N(\mathbb{R})\otimes_{\gamma_2}\ell_\infty^N(\mathbb{R})} \leq 1\Big\}.$$

To summarise, we have shown that the classical and the quantum bias of a given game G can be completely characterized by a tensor norm on a Banach space. Hence all the information about Alice and Bob winning a given game can be understood by computing the norm which is a geometrical quantity associated with the game G. Before ending this subsection, we shall recall the Theorem 3.2.23 of Grothendieck:

$$\beta^*(G) \leq K_G^{\mathbb{R}} \beta(G).$$

where the inequality above can be understood geometrically as $\mathbb{L}_N \subset \mathbb{Q}_N \subset K_G^{\mathbb{R}} \mathbb{L}_N$. The Grothendieck constant is a number that was discovered for purely mathematical reasons and now plays an important role in understanding the difference and the limitation between the classical and the quantum worlds.

In this Chapter, we have seen that nonlocality is described in the nonlocal game framework which is intrinsically linked to the theory of Banach spaces. We have shown that the classical value and the quantum value of a game can be completely described by tensor norms associated with a given game. The rate between the classical and quantum values of the game is bounded by the famous Grothendieck constant where for example in the CHSH game $K_{G_{\text{CHSH}}}^{\mathbb{R}} = \sqrt{2}$.

We will end this section by giving two recent results that are among the most important results at the intersection between nonlocality and Banach space theory. The first result that we shall introduce answers the question of what happens if we add more players, and the second result deals with a mathematical conjecture that was solved recently using techniques from quantum information theory; we shall present its historical origin and development.

All that we have presented in this Chapter is restricted to the two-player setting of Alice and Bob using dichotomic measurement apparatuses. One can ask *what happens if we consider more than two players.* The answer to this question is very surprising. It was shown [49] for the first time that if we only add another player Charlie, the ratio of the quantum bias over the classical bias is unbounded and it diverges with the number of questions. The proof in [49] uses techniques of *operator space theory* [50]. The divergence result was established also in [46, 51, 52]. In the following, we shall give the result in [46] where the authors have shown the existence of a 3−tensor T of dimension $N^2 \times N^2 \times N^2$, for which the ratio of the quantum bias $\beta^*(G_T)$ and the classical one $\beta(G_T)$ associated to a game G_T constructed from T *grows with the number of questions.*

Theorem 4.4.12. *[46] For any integer n and $N = 2^n$ there exists a three-player XOR game G_N with N^2 questions per player, such that*

$$\beta^*(G_N) \geq \Omega(\sqrt{N} \log^{-\frac{5}{2}} N)\beta(G_N).$$

Above, the symbol Ω denotes an asymptotic lower bound.

In a recent breakthrough, nonlocal games were used to establish the equality of two complexity classes: MIP*=RE [53]. This result (of 206 pages.) answers a very old conjecture of operator algebras that was asked by Alain Connes known as *Connes' embedding problem* (CEP). Several formulations of CEP were established (see [54] for a survey and the reference therein), we shall briefly mention here the equivalence of CEP with Kinchberg's conjecture[14]. It was shown in [57] that Kinchberg's conjecture is equivalent to Tsirelson's problem in the setting of nonlocal games. Tsirelson's problem asks briefly if *the quantum mechanical description of the physical reality of two space-like separated systems is it the same as if one uses finite or infinite dimensional Hilbert spaces.* We will recall more precisely in what follows Tsirelson's problem. This problem is central and fundamental for nonlocal games or the physics community which motivates our brief historical summary. To give a complete overview of the topic with all the results far beyond this thesis also it can easily take the same amount of pages of this PhD manuscript. Instead, we shall refer to [58] and the reference therein for an introduction to the MIP*=RE paper.

We shall start by giving the statement of the CEP, we shall recall some definitions that we have encountered in Section 2.5 and some results in operator algebras theory.

Definition 4.4.13. *A von Neumann algebra is a* bicommutant *sub-algebra $\tilde{\mathcal{A}} \subseteq \mathcal{B}(\mathcal{H})$:*

$$\tilde{\mathcal{A}}'' = \tilde{\mathcal{A}}.$$

where the commutant *of $\tilde{\mathcal{A}}$ is defined by $\tilde{\mathcal{A}}' := \{C \in \mathcal{B}(\mathcal{H}) | \forall A \in \tilde{\mathcal{A}} : \ AC = CA\}$ and the bicommutant is $(\tilde{\mathcal{A}}')'$.*

The motivation of this definition was established by von Neumann, where he showed the following theorem.

Theorem 4.4.14. *[17] Let $\mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$ be an algebra of operators such that $I \in \mathcal{A}$ and $X \in \mathcal{A} \implies X^* \in \mathcal{A}$. Then $\mathcal{A}'' = \bar{\mathcal{A}}^w = \bar{\mathcal{A}}^s = \bar{\mathcal{A}}^{wk^*}$[15].*

This fundamental result is due to von Neumann known as the bicommutant theorem. The importance of this theorem relies on the link between the algebraic property (bicommutant algebra) and topology. The classification of von Neumann algebras was initiated by von Neumann and Murray in [59] and ended with Alain Connes in [60]. However, a very important conjecture in operator algebras remained open known as the *Connes' embedding problem.*

The original problem is difficult to state here, however, there exists the simpler version of the problem known as the *matricial microstates problem*, see [58]. The matricial microstates

[14] Kinchberg conjecture asks if the *minimal and the maximal tensor product of two copies of C^*-algebra of a free group $\mathbb{F}_2$ are the same?* see [55] for the original formulation or [56, Appendix D].

[15] where $\bar{\mathcal{A}}^w, \bar{\mathcal{A}}^s, \bar{\mathcal{A}}^{wk^*}$ are the closure of $\mathcal{A}$ with respect to weak, strong and weak*- topology.

problem starts from the following definition: given a C^*-algebra $\mathcal{A}$ and tracial sate ω[16], n self-adjoint elements $h_1, \cdots, h_n \in \mathcal{A}$ of norm 1, an integer k and $\varepsilon > 0$, a (k, ε)-matricial microstate is a matrix algebra $\mathcal{M}_d$ and self-adjoint matrices $H_1, \cdots, H_n \in \mathcal{M}_d$ of norm 1, such that

$$|\omega(h_{i_1} \cdots h_{i_j}) - \omega_d(H_{i_1} \cdots H_{i_j})| < \varepsilon.$$

holds for every possible product of $j \leq k$ elements. When this is true for $(\mathcal{A}, \omega)$ for all n, k, ε, we say that $(\mathcal{A}, \omega)$ has *matricial microstates*. Informally the statement of this problem can be understood as approximating a tracial state on an abstract algebra $\mathcal{A}$ by a normalised trace of self-adjoint matrices in $\mathcal{M}_d$.

There exist several equivalent ways of introducing the CEP problem [54], one way that can be relevant in this thesis is its link with *the Tsirelson problem*. As we have described in this Chapter, nonlocality is described in the nonlocal game framework, where the players Alice and Bob use either classical or quantum strategies by sharing a quantum state living in the tensor product of Alice's and Bob's Hilbert space. However, there exists another description of quantum strategies by using quantum states living in *infinite dimensional Hilbert spaces*. To differentiate it from the tensor product strategy we shall denote it by ${\mathcal{Q}'}_{N,M}$.

Alice and Bob play the game with a referee and they share the same infinite-dimensional Hilbert space $\mathcal{H}$. The quantum strategies are now given by the quantum set ${\mathcal{Q}'}_{N,M}$, known as the *quantum commuting set*, defined by:

$${\mathcal{Q}'}_{N,M} := \left\{ \mathbb{P}_Q(a\,b|x\,y) \,\middle|\, \mathbb{P}_Q(a\,b|x\,y) = \langle\psi|\, A_{a|x}\, B_{b|y}\, |\psi\rangle \right\}$$

where Alice and Bob perform the measurement on the same Hilbert space $\mathcal{H}$ with their respective POVM given respectively by

$$\left\{ A_{a|x}, 1 \leq x \leq N, 1 \leq a \leq M \right\} \subseteq \mathbb{B}(\mathcal{H}) \quad \text{and} \quad \left\{ B_{b|y}, 1 \leq y \leq N, 1 \leq b \leq M \right\} \subseteq \mathbb{B}(\mathcal{H})$$

Moreover, the space-like separation of the two players means that their local measurement apparatus commute $[A_{a|x}, B_{b|y}] = 0$. Tsirelson has shown that if $N = M = 2$ the quantum commuting set and the quantum set are the same $\mathcal{Q}_{2,2} = \mathcal{Q}'_{2,2}$[17]. Trivially we have $\mathcal{Q}_{N,M} \subseteq \mathcal{Q}'_{N,M}$, this can be easily shown by $[A_{a|x} \otimes I_B, I_A \otimes B_{b|y}] = 0$. *The Tsirelson problem asks if* $\mathcal{Q}_{N,M} = \mathcal{Q}'_{N,M}$ *for all* N *and* M. It turns out that asking whether using commuting strategies or the tensor product strategies is equivalent to CEP [57]. The recent breakthrough paper MIP*=RE [53] shows that the quantum commuting set and the quantum probability set $\mathcal{Q}_{N,M}$ are not equivalent for all N and M by using tools and concepts from computer science. The Tsirelson problem is false, the quantum commuting set and the quantum set are hence different, and Connes conjecture is false.

[16]We say a state ω is tracial if $\forall x, y \in \mathcal{A}$ we have $\omega(xy) = \omega(yx)$ (see [17]).

[17]More precisely Tsirelson has shown that $\mathcal{Q}_{2,2} = \mathcal{Q}^s_{2,2} = \mathcal{Q}^a_{2,2} = \mathcal{Q}'_{2,2}$. $\mathcal{Q}^s_{N,M}$ stands for *quantum spatial*, where Alice and Bob may have different finite-dimensional Hilbert spaces, and $\mathcal{Q}^a_{N,M}$ for the *quantum approximate set*, which is defined as the closure of the quantum set.

Chapter 5

Compatibility of quantum measurements

One fundamental difference between quantum theory and classical physics is the existence of incompatible observables. Historically, the incompatibility of quantum measurements was understood as the non-commutativity of the observables, as introduced in the famous paper of Heisenberg [4]. The existence of non-commutative observables allowed him to show his well-known uncertainty principle. However, a new notion of incompatibility emerges with Busch and Lahti in [61, 62] where the commutativity of the observables is very particular and restrictive from the quantum informational point of view. Other types of compatibility have been explored recently, such as the compatibility of quantum channels [63, 64] or the compatibility of quantum instruments [65]; for more recent development, see the reviews [66, 67]. Nowadays, several works on understanding the (in)compatibility of quantum measurements become relevant for applications such as the link between the incompatibility of quantum measurements and Bell non locality[1] established in [9]. The violations of Bell inequalities are important for example in cryptography [45]. This chapter summarises some results obtained in [1] and gives an introduction to the notion of compatibility. At the end of the chapter, a new point of view on compatibility based on the tensor norms, is introduced and it will play later in Chapter 6 a crucial role to understand the link between measurement (in)compatibility and Bell inequality violations.

In Section 5.1 we will introduce the definition of compatibility and its formulation in terms of Semidefinite programs and we will end this section by introducing the compatibility dimension [1]. In Section 5.2 we will introduce different types of noise models and their link with approximate quantum cloning. We will end this chapter with Section 5.3, where we introduce new criteria for compatibility using the tensor norm framework that will become relevant in Chapter 6.

5.1 Definition and basic properties

As we have seen in Chapter 2, the measurements on a quantum system are generally described by POVMs (see Definition 2.2.1). One of the key differences between the classical world and the quantum one is the existence of incompatible measurements which describe measurements that we cannot perform at the same time. In the following section, we will introduce the notion of (in)compatibility of quantum measurements, we will see its formulation as an SDP. For a general discussion and extensive introduction to the topic see for example [66].

Definition 5.1.1. *Two POVMs $A = (A_1, \ldots, A_k)$, $B = (B_1, \ldots, B_l)$ on $\mathcal{M}_d$ are called* com-

[1] We will develop the link between the incompatibility of quantum measurements and Bell inequality violation in the Chapter 6.

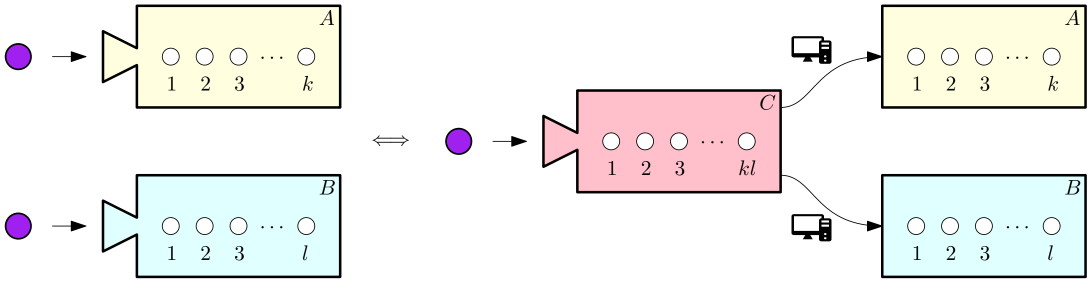


Figure 5.1: The joint measurement of A and B is simulated by by a third measurement C, followed by classical post-processing.

patible *if there exists a* joint POVM $C = (C_{11}, \ldots, C_{kl})$ *on* $\mathcal{M}_d$ *such that* A *and* B *are its respective* marginals*:*

$$\forall i \in [k], \qquad A_i = \sum_{j=1}^{l} C_{ij}.$$

$$\forall j \in [l], \qquad B_j = \sum_{i=1}^{k} C_{ij}.$$

More generally, a g-tuple of POVMs $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$ *is called compatible if there exists a POVM C with outcome set* $[k_1] \times \cdots \times [k_g]$ *such that, for all* $x \in [g]$*, the POVM* $A^{(x)}$ *is the x-th marginal of C:*

$$\forall i_x \in [k_x], \qquad A^{(x)}_{i_x} = \sum_{i_1=1}^{k_1} \cdots \sum_{i_{x-1}=1}^{k_{x-1}} \sum_{i_{x+1}=1}^{k_{x+1}} \cdots \sum_{i_g=1}^{k_g} C_{i_1 i_2 \cdots i_g}$$
$$= \sum_{\substack{\mathbf{j} \in [k_1] \times \cdots \times [k_g] \\ j_x = i_x}} C_{\mathbf{j}}.$$

As we have mentioned earlier, one can think of compatibility as commutativity but in general, is it the same only for the case when one uses projective measurement instead of POVMs. In the following proposition, we shall give the link between compatibility and commutativity.

Proposition 5.1.2. *[68, 69] Let A_i and B_j two observables on a Hilbert space, if A_i and B_j satisfies the following inequality:*

$$\|[A_i, B_j]\| \leq 4\|A_i - A_i^2\| \cdot \|B_j - B_j^2\|.$$

then A_i, and B_j are compatible. In particular if A_i and B_j are PVM's then they are compatible if and only if

$$[A_i, B_j] = 0. \quad \forall i, j.$$

Pictorially, one can think of the joint measurement apparatus C as a big box, where one can deduce all the measurements of the first POVM A and B (see Figure 9.3 for an illustration). Alternatively, we can define compatible measurements, as the measurements arising from a *post-processing* of a single POVM. The two definitions are equivalent, for more details see [66, Section 3.1].

Proposition 5.1.3. *An N-tuple of POVMs* $\mathbf{A} = (A^{(1)}, \ldots, A^{(N)})$ *is compatible if and only if there exists a joint POVM* $(C_k)_{k\in[K]}$ *and a family of conditional probabilities* $\big(p_x(\cdot|\cdot)\big)_{x\in[N]}$ *such that*

$$\forall x \in [N], \forall i \in [k_x], \qquad A^{(x)}_i = \sum_{k\in[K]} p_x(i|k) C_k.$$

Deciding whether tuples of measurements (POVMs) are compatible, is in general, a difficult task. However, compatibility by using a computational method called *semidefinite programing* or simply SDP. These are a type of convex optimization programs with positive constraints. For a general introduction to convex optimization see [70]. In the following, we will introduce the SDP formulation of compatibility in the case of POVMs with 2-outcomes which can be easily generalized for more outcomes [9].

Proposition 5.1.4. *[9, Proposition 1] Let two 2outcome POVMs $\{Q, I-Q\}$ and $\{P, I-P\}$, where P, Q are $d \times d$ self-adjoint matrices satisfying $0 \leq P, Q \leq I_d$. The pair of POVMs are compatible if and only if $\varepsilon_0 \leq 1$, where*

$$\varepsilon_0 := \inf \big\{ \varepsilon \,:\, \exists \delta \geq 0 \quad s.t. \quad \delta + I - Q - P \geq 0,\, Q + \varepsilon I - \delta \geq 0,\, P + \varepsilon I - \delta \geq 0 \big\}, \tag{5.1}$$

with $X \geq 0$ denotes the condition that X is a positive semidefinite matrix.

The above formula corresponds to the value of a semidefinite program encoding the existence of a joint measurement for the POVMs $\{P, I-P\}$ and $\{Q, I-Q\}$. Generally, every SDP comes with a dual formulation[2] given by:

Proposition 5.1.5. *[9] Given the above optimization problem for deciding compatibility, its dual formulation is given by:*

$$\varepsilon^* = \sup_{X,Y,Z \geq 0} \Big\{ \mathrm{Tr}[X(Q+P-I)] - \mathrm{Tr}[YQ] - \mathrm{Tr}[PZ] \; \textit{with}\; X \leq Y + Z,\; \mathrm{Tr}[Y+Z] = 1 \Big\},$$

Proof. Let us consider the following Lagrangian, corresponding the primal SDP (5.1).

$$\mathcal{L} := \varepsilon - \langle X, \delta + I - Q - P \rangle - \langle Y, \varepsilon I + Q - \delta \rangle - \langle Z, \varepsilon I + P - \delta \rangle - \langle C, \delta \rangle.$$

Above X, Y, Z, C are positive semidefinite matrices which represent the constraints of the primal optimization problem. Due to the strict feasibility of the SDP, we can compute its dual optimal value which is the same as the optimal one of the primal (see [9]) satisfying Slater's condition (see [70]). Thus, we have the following equality:

$$\inf_{\varepsilon,\delta} \sup_{X,Y,Z,C} \mathcal{L} = \sup_{X,Y,Z,C} \inf_{\varepsilon,\delta} \mathcal{L}.$$

A simple calculation shows that

$$\inf_{\varepsilon,\delta} \mathcal{L} = \langle X, Q + P - I \rangle - \langle Y, Q \rangle - \langle P, Z \rangle$$

with $\mathrm{Tr}[Y+Z] = 1$ and $Z + Y - X - C = 0 \iff X \leq Y + Z$, which is precisely the dual formulation from the statement. □

Several approaches and attempts to understand the compatibility of quantum measurements were explored in the literature, see [67] for a recent overview of the topic. The core article [1] is devoted to understanding the effect of the Hilbert space dimension on the compatibility of the quantum measurements. The notion of compatibility dimension was introduced and examples were analyzed in [1, Section 4].

[2] Generally the dual formulation of an SDP is not necessarily equal to its primal, however in the case of the compatibility they coincide due to the strict feasibility condition of the problem due to the positivity of the POVMs [9].

Definition 5.1.6. *[1, Definition 4.4.4] Given a g-tuple of POVMs* $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$ *on a d-dimensional quantum system, we define their* compatibility dimension *as the largest dimension r for which there* exists *an isometry* $V : \mathbb{C}^r \to \mathbb{C}^d$ *reducing the POVMs to a compatible g-tuple:*

$$R(\mathbf{A}) := \max\{r \in [d] \,:\, \exists V : \mathbb{C}^r \to \mathbb{C}^d \text{ isom. s.t. } V^*A^{(1)}V, \ldots, V^*A^{(g)}V \text{ are comp.}\}$$

Similarly, we define the strong compatibility dimension *of a g-tuple of POVMs* $\mathbf{A}$ *as the largest dimension r for which* all *isometries* $V : \mathbb{C}^r \to \mathbb{C}^d$ *reduce the POVMs to a compatible g-tuple:*

$$\bar{R}(\mathbf{A}) := \max\{r \in [d] \,:\, \forall V : \mathbb{C}^r \to \mathbb{C}^d \text{ isom., } V^*A^{(1)}V, \ldots, V^*A^{(g)}V \text{ are comp.}\}$$

Bounds on the compatibility dimension were obtained in [1] by using algebraic techniques. An application of the definition above is given in Theorem 5.2.5 in the following section 5.2. We should mention that other definitions of compatibility dimensions were introduced and explored in [71].

5.2 Noise models and cloning

In the following section, we will introduce different types of noise models that were considered in the literature. We have seen in Chapter 2 that it follows from the principles of quantum mechanics, that we cannot perfectly copy quantum states (this is known as the non-cloning theorem). However, there exists a way to make *aproximate copies* that we shall investigate in this section as well as its link with a noise model for quantum measurement. We will end this section with an application of the compatibility dimension for noisy PVMs constructed from *mutually unbiased basis* or simply MUB.

As we have described in the previous subsection, one of the fundamental differences between classical physics and quantum mechanics is the existence of incompatible measurements. However, there exists a procedure that makes incompatible measurements as compatible as we want. This is achieved by adding some classical *noise* to the POVMs. The noise is generally given by a parameter $t \in [0, 1]$ that will mix the original POVMs with a *trivial measurement operator*. Intuitively, the more the parameter t grows the more POVMs become compatible. In the following, we will introduce different *noise model* that were established (see [72] for an extensive review). Also, we will introduce the connection between one of the noise models with the asymmetric cloning problem, which is a way to go around the non-cloning theorem.

The first type of noise model that we will introduce is the *white noise*. Instead of measuring the POVM $A_1, \cdots, A_N$, one measures the noisy POVM $A'_1, \cdots, A'_N$ given by a convex combination of $A_1, \cdots, A_N$ and the I_d with some parameter $t \in [0, 1]$:

$$A_i \to A'_i := t\,A_i + (1-t)\frac{I_d}{N}, \quad i \in [N].$$

The new POVM $(A'_1, \cdots, A'_N)$ corresponds to a device that performs the original measurement with probability t and with probability $1-t$ outputs an outcome uniformly at random. Actually the POVM $\mathcal{I} := (\frac{I_d}{N}, \cdots, \frac{I_d}{N})$ is a trivial POVM: the measurements of $\mathcal{I}$ produces the same outcome statistics for every quantum state. Other classes of trivial operators can be made where the POVM $\mathcal{I}$ is a special one. The class of trivial POVMs are of the form $E = (e_1 I_d, \cdots, e_N\, I_d)$ with $e := (e_1, \cdots, e_N)$ is a probability distribution, where the choice of a given distribution specifies completely the type of noise model. Another type of noise model that was also considered in the literature (see [72]) is given by $e := (\mathrm{Tr}[A_1]/d \cdot I_d, \cdots, \mathrm{Tr}[A_N]/d \cdot I_d)$ that depends on the initial POVM itself

$$A_i \to A'_i := t\,A_i + (1-t)\frac{\mathrm{Tr}[A_i]}{d}\,I_d, \quad i \in [N].$$

One can see immediately that this noise model is linear in A_i.

Remark 5.2.1. *In all this description, we have only considered one noise parameter t. One can actually study those different types of noise models with a vector of parameters $\mathbf{t} := (t_1, \cdots, t_g) \in [0,1]^g$ when cloning g-tuples of POVMs*

In what follows, we will introduce the connection between noise models and *approximate quantum cloning*. We recall from Chapter 2 that the non-cloning theorem is one of the key concepts that differentiate the classical world from the quantum one. As we have seen, technically one cannot construct a quantum channel $\Phi : \mathcal{M}_d \to \mathcal{M}_d^{\otimes g}$ with the property that

$$\forall \rho \in \mathcal{M}_d^{1,+}, \forall j \in \{1, \cdots, g\}, \qquad \operatorname{Tr}_{[g]\setminus\{j\}} \Phi(\rho) = \rho.$$

Werner and Keyl in [73, 74] initiated a way to go around this obstruction by introducing what is known as *the symmetric approximate cloning*, and later generalized to *asymmetric quantum cloning* case [75–77] and more recently in [78]. Approximate quantum cloning (symmetric or asymmetric) characterizes an imperfect cloning machine, where its role is to produce imperfect clones (copies) for arbitrary input quantum states. The imperfection relies on the fact we act with the quantum channel, and by taking the marginals of the channel we obtain a noisy residual state described by a convex combination of the initial state and a trivial operator with some parameter $t \in [0,1]$. The asymmetric quantum cloning machine is characterized by a tuple of parameters $t_i \in [0,1]^g$ and the symmetric case reduces to a single parameter $t \in [0,1]$. Formally we have the following definition of asymmetric quantum cloning:

Definition 5.2.2. *The approximation parameters of physical $1 \to g$[3] asymmetric cloners on $\mathbb{C}^d$ are described by the following set:*

$$\Gamma^{clone}(g,d) := \Big\{\mathbf{t} \in [0,1]^g \; : \; \exists \Phi : \mathcal{M}_d \to \mathcal{M}_d^{\otimes g} \text{ quantum channel such that}$$
$$\forall \rho \in \mathcal{M}_d, \forall j \in [g], \quad \operatorname{Tr}_{[g]\setminus\{j\}} \Phi(\rho) = t_j \rho + (1-t_j)\frac{I_d}{d}\Big\}.$$

The task of cloning quantum states can be reinterpreted in the Heisenberg picture of quantum mechanics by looking at the dual map of a channel; this operation acts naturally on quantum measurements. In this picture, the dual property of producing imperfect clones is having noisy measurements. Let us define the asymmetric dual map for the POVMs and the corresponding set of cloning parameters.

Consider the set of parameters for these dual maps:

$$\tilde{\Gamma}^{clone}(g,d) := \Big\{\mathbf{t} \in [0,1]^g \; : \; \exists \Psi : \mathcal{M}_d^{\otimes g} \to \mathcal{M}_d \text{ unital and completely positive such that}$$
$$\forall X \in \mathcal{M}_d, \forall j \in [g], \quad \Psi(I^{\otimes(j-1)} \otimes X \otimes I^{\otimes(g-j)}) = t_j X + (1-t_j)\frac{\operatorname{Tr} X}{d} I\Big\}.$$

We have the following proposition, that shows the direct link between the asymmetric cloning set and its dual version.

Proposition 5.2.3. *[1, Proposition 4.3.4] The dual and the primal sets of cloning parameters are identical: $\forall g, d \geq 2$,*

$$\tilde{\Gamma}^{clone}(g,d) = \Gamma^{clone}(g,d).$$

There is a connection between the compatibility of POVMs and the approximate quantum cloning devices: it was shown in [1] that if the parameters $\mathbf{s} \in \Gamma^{clone}(g,d)$ then a g-tuple of POVMs are compatibly generalising the result in [79, Proposition III.3] to the case of more than two POVMs.

[3] This notation denotes g copies obtained from one quantum state.

Theorem 5.2.4. *[1, Theorem 4.3.6] Let $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$ be a g-tuple of POVMs on $\mathcal{M}_d$ having, respectively, $k_1, \ldots, k_g$ outcomes. Define, for all $x \in [g]$,*

$$s_x := 1 - \min_{i \in [k_x]} \frac{d\lambda_{\min}(A_i^{(x)})}{\operatorname{Tr} A_i^{(x)}} \in [0,1].$$

where $\lambda_{min}(\cdot)$denotes the smallest eigenvalue of an operator. If $\mathbf{s} \in \Gamma^{clone}(g,d)$, then the POVMs in $\mathbf{A}$ are compatible.

Other approaches were established to understand the link between compatibility and asymmetric quantum cloning by using tools from free spectrahedra theory in [80, 81].

In the following, we will introduce an application of compatibility dimension introduced in [1]: if we consider incompatible noisy POVMs constructed from MUBs, we can find a Hilbert space of smaller dimension on which they become compatible. For that we shall recall that a set of g orthonormal bases $\left\{\{|b_i^{(x)}\rangle\}_{i\in[d]}\right\}_{x\in[g]}$ are called *mutually unbiased* (MUB) [82, 83] if

$$\forall x \neq y \in [g],\, \forall i,j \in [d], \qquad |\langle b_i^{(x)} | b_j^{(y)}\rangle|^2 = \frac{1}{d}.$$

Consider two mutually unbiased bases $\{|a_1\rangle, \ldots, |a_d\rangle\}$ and $\{|b_1\rangle, \ldots, |b_d\rangle\}$ in $\mathbb{C}^d$. Let us introduce the noisy versions of the POVMs A and B.

$$\mathcal{N}_\lambda[A] = \left(\lambda|a_1\rangle\langle a_1| + (1-\lambda)\frac{I_d}{d}, \ldots, \lambda|a_d\rangle\langle a_d| + (1-\lambda)\frac{I_d}{d}\right)$$
$$\mathcal{N}_\mu[B] = \left(\mu|b_1\rangle\langle b_1| + (1-\mu)\frac{I_d}{d}, \ldots, \mu|b_d\rangle\langle b_d| + (1-\mu)\frac{I_d}{d}\right).$$

The values (λ, μ) for which the POVMs above are compatible have been computed in [84, 85]: for $(\lambda,\mu) \in [0,1]^2$, $\mathcal{N}_\lambda[A]$ and $\mathcal{N}_\mu[B]$ are compatible iff

$$\lambda + \mu \leq 1 \text{ or } \lambda^2 + \mu^2 + \frac{2(d-2)}{d}(1-\lambda)(1-\mu) \leq 1.$$

We consider first the symmetric case $\lambda = \mu$. In this situation, the POVMs $\mathcal{N}_\lambda[A]$ and $\mathcal{N}_\lambda[B]$ are compatible if and only if

$$\lambda \leq \frac{1}{2}\left(1 + \frac{1}{1+\sqrt{d}}\right). \tag{5.2}$$

We shall show that for the same symmetric amount of noise and with a particular choice of an isometry $V : \mathbb{C}^r \to \mathbb{C}^d$, reducing the dimension of two incompatible noisy MUB measurements renders them compatible.

Theorem 5.2.5. *[1, Theorem 4.7.1] Consider two POVMs A, B corresponding to a pair of mutually unbiased bases which can be extended to a triple of MUBs. For any $2 \leq r < \sqrt{d}$, there exists a non-empty interval $\Lambda_{r,d} \subset [0,1]$ such that, for all $\lambda \in \Lambda_{r,d}$,*

- *the noisy MUB measurements $\mathcal{N}_\lambda[A]$, $\mathcal{N}_\lambda[B]$ are incompatible*
- *their reduced versions $V^*\mathcal{N}_\lambda[A]V$, $V^*\mathcal{N}_\lambda[B]V$ are compatible,*

where $V : \mathbb{C}^r \to \mathbb{C}^d$ is an isometry obtained by truncating a third MUB given by $V := \sum_{k=1}^r |c_k\rangle\langle k|$ and

$$\Lambda_{r,d} := \left(\frac{2+\sqrt{d}}{2(1+\sqrt{d})}, \frac{2+r}{2(1+r)}\right]$$

To summarize, restricting a pair of incompatible POVMs to a subspace of $\mathbb{C}^d$ of dimension r renders them compatible. The previous theorem emphasizes the fact that the Hilbert space dimension plays a similar role to the amount of noise present in a POVM: reducing it (respectively increasing noise parameters) makes the POVMs "more compatible".

5.3 Compatibility via tensor norms

In this section, we will introduce a new approach to compatibility based on a tensor norm known as the *compatibility norm*. We will mention that the compatibility norm is a tensor norm in the sense of Grothendieck and we will give a characterization of compatibility using this tensor norm. We will end this section with a result relating to the white noise model and the compatibility tensor norm.

The compatibility norm will play an important role to understand the link between incompatibility and nonlocality in Chapter 6. This approach was first established in the setting of GPT[4] in [86]. If we reduce it in the case of quantum theory the compatibility norm is defined as follows [87, 88].

Definition 5.3.1. *[87, Definition 3.1][The compatibility tensor norm] For a tensor $A \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$, we define the following quantity:*

$$\|A\|_c := \inf \left\{ \Big\| \sum_{j=1}^{K} H_j \Big\|_\infty \ : \ A = \sum_{j=1}^{K} z_j \otimes H_j, \ \ s.t. \ \ \forall j \in [K], \, \|z_j\|_\infty \leq 1 \text{ and } H_j \geq 0 \right\}.$$

Remark 5.3.2. *In [86], the definition of the compatibility (tensor) norm is* only *valid in the case of POVMs with two outcomes.*

The quantity $\|\cdot\|_c$ is a tensor norm on $(\mathbb{R}^N, \|\cdot\|_\infty) \otimes (\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$ where we recall from Chapter 3 that $\|\cdot\|_c$ should satisfy the following conditions

$$\|A\|_{\ell_\infty^N(\mathbb{R}) \otimes_\varepsilon \mathcal{S}_\infty^d(\mathbb{C})} \leq \|A\|_c \leq \|A\|_{\ell_\infty^N(\mathbb{R}) \otimes_\pi \mathcal{S}_\infty^d(\mathbb{C})}$$

with the injective norm and projective norm on $\ell_\infty^N(\mathbb{R}) \otimes \mathcal{S}_\infty^d(\mathbb{C})$ are respectively given by:

$$\|A\|_{\ell_\infty^N(\mathbb{R}) \otimes_\varepsilon \mathcal{S}_\infty^d(\mathbb{C})} := \sup \Big\{ \langle x \otimes Y, A \rangle, \|x\|_{\ell_1^N(\mathbb{R})} \leq 1, \|Y\|_{\mathcal{S}_1^d(\mathbb{C})} \leq 1 \Big\}.$$

and

$$\|A\|_{\ell_\infty^N(\mathbb{R}) \otimes_\pi \mathcal{S}_\infty^d(\mathbb{C})} := \inf \Big\{ \sum_i \|x_i\|_{\ell_1^N(\mathbb{R})} \|Y_i\|_{\mathcal{S}_1^d(\mathbb{C})}; \, A = \sum_i x_i \otimes Y_i \Big\}.$$

more precisely we have the following proposition.

Proposition 5.3.3. *[87, Proposition 3.3]*
The $\|\cdot\|_c$ quantity is a tensor norm on $(\mathbb{R}^N, \|\cdot\|_\infty) \otimes (\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$.

The tensor norm $\|\cdot\|_c$ characterizes if the measurements with two outcomes are compatible. The following theorem encodes in a geometrical sense the compatibility of the POVMs.

Theorem 5.3.4. *[86, Theorem 9.2] Let $A = (A_1, \dots, A_N)$ be a N-tuple of self-adjoint $d \times d$ complex matrices. Then:*

1. *A is a collection of dichotomic quantum observables (i.e. $\|A_i\|_\infty \leq 1 \ \forall i$) if and only if $\|A\|_\varepsilon \leq 1$, where $\|\cdot\|_\varepsilon$ is the $\ell_\infty^N \otimes_\varepsilon S_\infty^d$ tensor norm.*
2. *A is a collection of* compatible *dichotomic quantum observables if and only if $\|A\|_c \leq 1$.*

As we have seen in the previous Section 5.2, even if the POVMs are not compatible one can add noise and ask for the minimal amount of noise needed to make them compatible, and how it is related to the norm $\|\cdot\|_c$.

[4]See Section 2.5 for an introduction.

Definition 5.3.5. *[2, Definition 5.3.7] For two (binary) measurements $\mathcal{P}$, $\mathcal{Q}$, we define their* noise compatibility threshold *as:*

$$\Gamma(P, Q) := \sup \left\{ \eta \in [0,1] \, : \, \mathcal{P}^\eta, \mathcal{Q}^\eta \text{ are compatible } \right\}.$$

with $\mathcal{P}^\eta = \eta(P, I-P) + (1-\eta)(I/2, I/2)$ and the same for $\mathcal{Q}^\eta$.

The following proposition will give the relation between the noise compatibility threshold and the compatibility tensor norm for the case of N measurement with two outcomes.

Proposition 5.3.6. *[2, Proposition 5.5.12] For any N-tuple of dichotomic observables $A = (A_1, A_2, \ldots, A_N) \neq 0$,*

$$\Gamma(A) = \frac{1}{\|A\|_c}.$$

The compatibility norm $\|\cdot\|_c$ allows quantifying in a geometrical manner the compatibility of quantum devices. We will see in Chapter 6 that this formulation of compatibility is very relevant to give a unified framework with nonlocality in the setting of nonlocal games, where we will introduce another relevant norm $\|\cdot\|_G$ that will characterize nonlocality. By comparing $\|\cdot\|_c$ and $\|\cdot\|_G$ we will understand if the incompatibility of quantum measurement is equivalent to the violation of the Bell inequality corresponding to the game G.

Chapter 6

Incompatibility and nonlocality

The incompatibility of quantum measurements and Bell inequality violations are the two fundamental concepts that differentiate quantum theory from the classical description of physical reality. It is well known that in order to observe a Bell inequality violation one needs to use incompatible measurements. In [9] the authors have shown that using incompatible measurements implies Bell inequality violation, hence incompatibility is equivalent to Bell inequality violation. However, this result is only valid for the CHSH game. In general, it was conjectured that incompatibility and Bell inequality violations are not equivalent [89, 90].

In this chapter, we will give a common framework for analyzing measurement incompatibility and nonlocality. For that, we will consider Alice and Bob playing a nonlocal game, where Alice's measurements are fixed. If her measurements are incompatible, she wants to know if she is violating any Bell inequality. For that, she will compute two tensor norms $\|\cdot\|_c$ and $\|\cdot\|_G$ of a tensor constructed from her measurement devices. Understanding the link between incompatibility and nonlocality is translated in this framework by comparing the two norms $\|\cdot\|c$ and $\|\cdot\|_G$. The result in [9], is that for the CHSH game, incompatibility is equivalent to the Bell inequality violation, which means in our framework that $\|\cdot\|_{G_{CHSH}} = \|\cdot\|_c$. The question remained open for general games G.

In Section 6.1 we will introduce the general framework for a unified description of incompatibility and nonlocality, where Alice's measurements are fixed. In Section 6.2 we will introduce the notion of $G-$Bell-(non)locality. This notion will characterize the observed nonlocal effect on Alice's side for fixed measurements. Mathematically, it is encoded by $\|\cdot\|_G$ and we will show that it is a tensor norm. In Section 6.3 we will give the main results, where we will compare the two norms $\|\cdot\|_G$ and $\|\cdot\|_c$ and we will show that if we want a strong equivalence between incompatibility and Bell inequality violation in the spirit of [9], the only game satisfying it is the CHSH game.

6.1 General framework

In this section, we will introduce a general framework to unify the incompatibility of quantum measurement and Bell-inequality violations. The framework of nonlocal games, particularly XOR games, was introduced in Chapter 4. In the usual setting, Alice and Bob are playing against a Referee and they want to maximize their winning probability either by using classical or quantum correlations. This is given respectively by the classical bias $\beta(G)$ and the quantum bias $\beta^*(G)$, where we recall that the quantum bias is computed as an optimization on Alice's and Bob's measurements, and on the shared quantum state (see Definition 4.4.2). It was shown in [9] that the incompatibility of two 2outcome quantum measurements is equivalent to the violation of the inequality corresponding to the CHSH game. In this section, we shall answer the following question: *Are there any other games allowing this equivalence with more than* 2 *binary measurements?*

To unify the incompatibility of quantum measurements and Bell inequality violations, we shall use the natural framework of nonlocal games and particularly the XOR games generalizing the CHSH one. But instead of optimizing on Alice's measurement for the quantum bias, we shall assume in this setting that *Alice measurements are fixed.* For a given game G, she asks *if she is violating any Bell inequality with incompatible measurements* (see Figure 6.1 for a representation of the thought experiment). To unify two fundamental notions of quantum theory, the *measurement incompatibility* and *Bell inequality violations*, we will consider the setting of *nonlocal XOR games*, where the rules of a correlation game are encoded in a real $N \times N$ matrix G, and *Alice's dichotomic measurements are fixed*, mathematically encoded by POVMs.

The maximum value of the game G, when Alice's measurements are fixed, is given by the following quantity.

Definition 6.1.1. *[2, Definition 5.6.1][The G-Bell-locality tensor norm] Let G an invertible* Bell functional *and Alice's N-tuple of dichotomic measurements $A = (A_1, \dots, A_N)$, we define the following tensor norm:*

$$\|A\|_G := \sup_{\|\psi\|=1} \sup_{\|B_y\| \leq 1} \left\langle \psi \middle| \sum_{x,y=1}^{N} G_{xy}\, A_x \otimes B_y \middle| \psi \right\rangle = \lambda_{\max} \left[\sum_{y=1}^{N} \left| \sum_{x=1}^{N} G_{xy}\, A_x \right| \right].$$

The quantity $\|A\|_G$ is the maximum value of the game G, when optimizing over quantum strategies, with Alice's measurements being fixed.

Definition 6.1.2. *[2, Definition 5.6.3] The measurements $A = (A_1, \dots, A_N)$ are called G-*Bell-local *if there is no violation of the Bell inequality corresponding to G:*

$$\|A\|_G \leq \beta(G).$$

with $\beta(G)$ is the classical bias *of the game. If this is not the case, we call Alice's measurements* G-Bell-nonlocal.

Regarding compatibility, we are concerned with the same question as before: are Alice's dichotomic measurements compatible or not? We will use the compatibility formulation with tensor norms (see Definition 5.3.1), that we recall for the reader's convenience.

Definition 6.1.3. *[87, Definition 3.1][The compatibility tensor norm] For a tensor $A \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$, we define the following quantity:*

$$\|A\|_c := \inf \left\{ \Big\| \sum_{j=1}^{K} H_j \Big\|_\infty \ : \ A = \sum_{j=1}^{K} z_j \otimes H_j, \ \ s.t. \ \ \forall j \in [K], \ \|z_j\|_\infty \leq 1 \ and \ H_j \geq 0 \right\}.$$

The infimum above is taken over all the possible decomposition of A as sums of K simple tensors, with K being an arbitrary finite integer.

The compatibility norm, together with the injective tensor product of ℓ_∞ and S_∞ norms, completely characterize the compatibility of tuples of dichotomic quantum measurements [86, 87]. We shall also recall from Section 5 that the compatibility norm $\| \cdot \|_c$ is a tensor norm in the following sense:

$$\|A\|_{\ell_\infty^N(\mathbb{R}) \otimes_\varepsilon \mathcal{S}_\infty^d(\mathbb{C})} \leq \|A\|_c \leq \|A\|_{\ell_\infty^N(\mathbb{R}) \otimes_\pi \mathcal{S}_\infty^d(\mathbb{C})}$$

where the injective and the projective norm are given respectively by:

$$\|A\|_{\ell_\infty^N(\mathbb{R}) \otimes_\varepsilon \mathcal{S}_\infty^d(\mathbb{C})} = \sup \left\{ \langle x \otimes Y, A \rangle, \|x\|_{\ell_1^N(\mathbb{R})} \leq 1, \|Y\|_{\mathcal{S}_1^d(\mathbb{C})} \leq 1 \right\}$$

and

$$\|A\|_{\ell_\infty^N(\mathbb{R}) \otimes_\pi \mathcal{S}_\infty^d(\mathbb{C})} = \inf \left\{ \sum_i \|x_i\|_{\ell_1^N(\mathbb{R})} \, \|Y_i\|_{\mathcal{S}_1^d(\mathbb{C})}; \ A = \sum_i x_i \otimes Y_i \right\}.$$

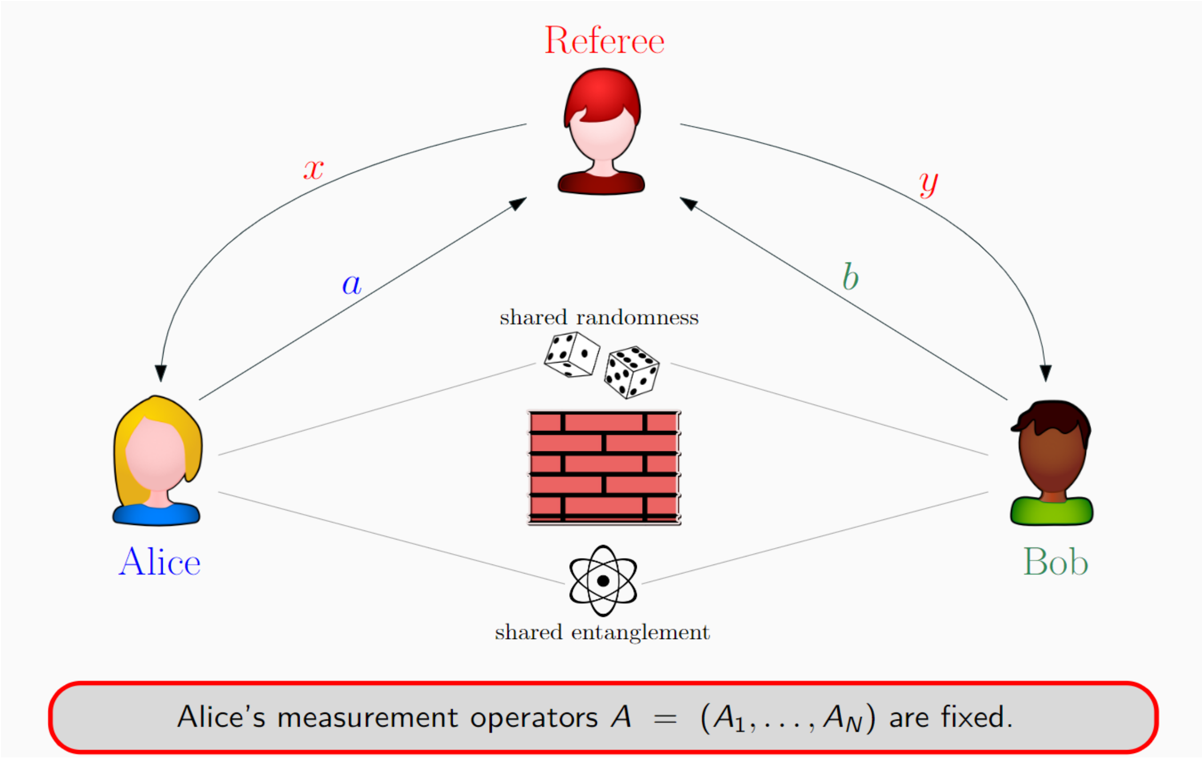


Figure 6.1: Alice and Bob playing the XOR game with Alice's measurements are fixed.

Proposition 6.1.4. *[87, Proposition 3.3] Let $A = (A_1, \dots, A_N)$ be a N-tuple of self-adjoint $d \times d$ complex matrices. Then:*

1. *A is a collection of dichotomic quantum observables (i.e. $\|A_i\|_\infty \le 1$ $\forall i$) if and only if $\|A\|_\varepsilon \le 1$.*
2. *A is a collection of* compatible *dichotomic quantum observables if and only if $\|A\|_c \le 1$.*

The compatibility norm allows Alice to know whether her measurements are compatible ($\|A\|_c \le 1$) or not ($\|A\|_c > 1$); in the latter case, the minimal quantity of white noise that needs to be mixed in the measurements in order to render them compatible is $1/\|A\|_c$, providing an operational interpretation of the compatibility norm.

To sum up, in the setting of tensor norms,

- Alice's measurements are *G-Bell-local* if and only if $\|A\|_G \le \beta(G) = \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N}$.
- Alice's measurements are compatible if and only if $\|A\|_c \le 1$.

In the rest of this section, we shall introduce the precise constructions, definitions, and theorems associated with the notion of G-Bell-locality in Section 6.2. In the last Section 6.3 we will compare the two norms $\|\cdot\|_c$ and $\|\cdot\|_G$ to understand if the incompatibility of the quantum measurements, equivalent to Bell inequality violation; mathematically this corresponds to $\|\cdot\|_c = \|\cdot\|_G$.

6.2 G-Bell-(non)locality

In the following section, we will introduce the fundamental notion of G-Bell-locality, which will play an important role to unify the concept of incompatibility and Bell inequality violation in the setting of nonlocal games. For that, we shall assume that Alice's measurements are fixed and she will calculate $\|\cdot\|_G$. We will see that this quantity is obtained by optimization over Bob's measurements and all the shared quantum state. The main result of this section is that the norm $\|\cdot\|_G$ is a tensor norm.

As we have described in Chapter 4, an XOR game is completely characterized by a $N \times N$ real matrix G. If Alice wants to know if she is violating any Bell inequality she will calculate the following norm $\|A\|_G$ associated with the N-tuple of her measurements apparatuses $A = (A_1, \cdots, A_N) \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$ where $A_x = A_{0|x} - A_{1|x}$ is the observable corresponding to the POVM $(A_{0|x}, A_{1|x})$.

Definition 6.2.1. *[2, Definition 5.6.3] Consider a fixed N-input, 2-outcome nonlocal game $G \in \mathcal{M}_N(\mathbb{R})$. Fix also Alice's measurements, a N-tuple of binary observables $A = (A_1, \ldots, A_N) \in \mathcal{M}_d^{sa}(\mathbb{C})^N$. The largest quantum bias of the game G, with Alice using the observable A_x to answer question $x \in [N]$, is given by*

$$\|A\|_G := \sup_{\|\psi\|=1} \sup_{\|B_y\| \leq 1} \left\langle \psi \middle| \sum_{x,y=1}^{N} G_{xy}\, A_x \otimes B_y \middle| \psi \right\rangle,$$

where the suprema are taken over bipartite pure states $\psi \in \mathbb{C}^d \otimes \mathbb{C}^D$ and over Bob's observables $B = (B_1, \ldots, B_N) \in \mathcal{M}_D^{sa}(\mathbb{C})^N$, where D is a free dimension parameter.

Definition 6.2.2. *[2, Definition 5.6.3] Given a nonlocal game G, we say that Alice's measurements $A = (A_1, \ldots, A_N)$ are G-*Bell-local *if for any choice of Bob's observables B and for any shared state ψ, one cannot violate the Bell inequality corresponding to G:*

$$\|A\|_G \leq \beta(G).$$

*If this is not the case, we call Alice's measurements G-*Bell-nonlocal.

The physical intuition behind the definition above is that no matter the optimization overall of Bob's measurements and all shared quantum states if Alice cannot do better than the classical bias $\beta(G)$ then her measurements are local.

Lemma 6.2.3. *[2, Lemma 5.6.4] Given a quantum game $(G_{xy})_{\{x,y=1\}}^N$ we can characterise the following equivalent formulation of $\|A\|_G$:*

$$\|A\|_G = \lambda_{\max} \left[\sum_{y=1}^{N} \left| \sum_{x=1}^{N} G_{xy}\, A_x \right| \right].$$

In the above lemma, we have a simpler equivalent definition of the (tensor) norm $\|A\|_G$.

Remark 6.2.4. *In Definition 6.1.1, the dimension of Alice's measurements is fixed (d), while the dimension of Bob's Hilbert space (D) is free. In the following, we will show that one can assume, without loss of generality, that Alice and Bob have Hilbert spaces of the same dimension ($D = d$ suffices in the optimization problem).*

Let us consider $D \geq d$, a quantum state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^D$, and N binary measurement operators $B_1, \ldots, B_N \in \mathcal{M}_D^{sa}(\mathbb{C})$. The idea is that the Schmidt decomposition of the bipartite pure quantum state $|\psi\rangle$ will induce a reduction of the effective dimension of Bob's Hilbert space from D to d. We start from the Schmidt decomposition of $|\psi\rangle$

$$|\psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i}\, |a_i\rangle \otimes |b_i\rangle .$$

Note that in the equation above, the number of terms is bounded by the smallest of the two dimensions, that is d. The orthonormal family $\{|b_i\rangle\}_{i\in[d]}$ spans a subspace of dimension d inside $\mathbb{C}^D$. Consider an arbitrary orthonormal basis $\{|\tilde{b}_i\rangle\}_{i\in[d]}$ *of $\mathbb{C}^d$ and the isometry*

$$V : \mathbb{C}^d \to \mathbb{C}^D \quad \text{such that} \quad \forall i \in [d], \quad V\,|\tilde{b}_i\rangle = |b_i\rangle\,.$$

Let us now introduce the quantum state

$$\mathbb{C}^d \otimes \mathbb{C}^d \ni |\tilde{\psi}\rangle := \sum_{i=1}^{d} \sqrt{\lambda_i}\,|a_i\rangle \otimes |\tilde{b}_i\rangle$$

and the measurement operators

$$\mathcal{M}_d^{sa}(\mathbb{C}) \ni \tilde{B}_y := V^* B_y V, \quad \forall y \in [N].$$

The normalization of the state and the fact that the $\tilde{B}_y$ are contractions follow from the isometry property of the operator V. We now have

$$\begin{aligned}\Big\langle\psi\Big|\sum_{x,y=1}^{N} G_{xy}\,A_x\otimes B_y\Big|\psi\Big\rangle &= \sum_{x,y=1}^{N} G_{xy}\sum_{i,j=1}^{d}\sqrt{\lambda_i\lambda_j}\langle a_i|A_x|a_j\rangle\underbrace{\langle b_i|B_y|b_j\rangle}_{=\langle\tilde{b}_i|V^*B_yV|\tilde{b}_j\rangle}\\ &= \sum_{x,y=1}^{N} G_{xy}\sum_{i,j=1}^{d}\sqrt{\lambda_i\lambda_j}\langle a_i|A_x|a_j\rangle\langle\tilde{b}_i|\tilde{B}_y|\tilde{b}_j\rangle\\ &= \Big\langle\tilde{\psi}\Big|\sum_{x,y=1}^{N} G_{xy}\,A_x\otimes\tilde{B}_y\Big|\tilde{\psi}\Big\rangle.\end{aligned}$$

The above computation shows that any correlation that can be obtained with Bob's Hilbert space of dimension D can also be obtained with a Hilbert space of dimension d, equal to that of Alice.

The main result of this section is that the norm $\|A\|_G$ is actually a tensor norm in the sense of the Definition 3.2.17, where we have that:

$$\|A\|_{\mathbb{R}^N\otimes_\varepsilon\mathcal{M}_d^{sa}(\mathbb{C})} \le \|A\|_G \le \|A\|_{\mathbb{R}^N\otimes_\pi\mathcal{M}_d^{sa}(\mathbb{C})}$$

with $(\mathbb{R}^N, \|\cdot\|_G)$ and $(\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$.

We shall endow the (real) vector spaces $\mathbb{R}^N$ and $\mathcal{M}_d^{sa}(\mathbb{C})$ with their respective norm $\|\cdot\|_G$ and the operator norm (or the Schatten-∞ norm, $\mathcal{S}_\infty$). Note that there is an abuse of notation here: we shall use $\|\cdot\|_G$ to denote norms on $\mathbb{R}^N$ and on $\mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$; the situation will be clear from the context. We shall now investigate the properties of the $\|\cdot\|_G$ norm with respect to this tensor product structure. We will consider that for given N-tuple of observables $(A_1, A_2, \ldots, A_N)$, we associate the tensor

$$A := \sum_{x=1}^{N} e_x \otimes A_x \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C}).$$

Definition 6.2.5. *[2, Definition 5.6.6] Given $p \in \mathbb{R}^N$, we define the following quantity:*

$$\|p\|_G := \sum_{y=1}^{N}\left|\sum_{x=1}^{N} G_{xy}\,p_x\right| = \|G^\top p\|_1.$$

Lemma 6.2.6. *[2, Lemma 5.6.7] Given an invertible matrix G, the function $\mathbb{R}^N \ni p \mapsto \|p\|_G$ is a norm.*

By the Lemma 6.2.6, we endow $\mathbb{R}^N$ with the norm $\|\cdot\|_G$, obtaining a Banach space $(\mathbb{R}^N, \|\cdot\|_G)$. In the following, we shall investigate the dual space of $(\mathbb{R}^N, \|\cdot\|_G)$. For that, we shall compute the dual norm of $\|\cdot\|_G$ denoted by $\|\cdot\|_G^*$.

Proposition 6.2.7. *[2, Proposition 5.6.8] The dual norm $\|\cdot\|_G^*$ is given by:*

$$\forall p \in \mathbb{R}^N, \qquad \|p\|_G^* = \max_y \Big| \sum_{z=1}^{N} (G^{-1})_{yz}\, p_z \Big| = \|G^{-1}p\|_\infty.$$

In the following proposition, we will give the factorization property that all tensor norms should satisfy if one considers tensors of rank one.

Proposition 6.2.8. *[2, Proposition 5.6.9] Given $A \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$ with $\mathbb{R}^N$ and $M_d^{sa}(\mathbb{C})$ are endowed with $\|\cdot\|_G$ and the natural operator norm respectively. Given the particular decomposition $A = p \otimes B$ with $p \in (\mathbb{R}^N, \|\cdot\|_G)$ and $B \in (\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$, one has*

$$\|p \otimes B\|_G = \|p\|_G \|B\|_\infty.$$

Now we are ready to give the main statement of the theorem in this section.

Theorem 6.2.9. *[2, Theorem 5.6.10] For a fixed N-input, a 2-output invertible nonlocal game G, the quantity $\|\cdot\|_G$ introduced in Definition 6.1.1, which characterizes the largest quantum bias of the game G when one fixes Alice's dichotomic measurements, is a reasonable crossnorm on $\mathcal{M}_d^{sa}(\mathbb{C})^N \cong \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$:*

$$\|A\|_{\mathbb{R}^N \otimes_\varepsilon \mathcal{M}_d^{sa}(\mathbb{C})} \le \|A\|_G \le \|A\|_{\mathbb{R}^N \otimes_\pi \mathcal{M}_d^{sa}(\mathbb{C})}$$

with $(\mathbb{R}^N, \|\cdot\|_G)$ and $(\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$.

Where we recall the definitions of the projective and the injective norms in this setting:

$$\|A\|_{\mathbb{R}^N \otimes_\pi \mathcal{M}_d^{sa}(\mathbb{C})} := \inf \Big\{ \sum_{i=1}^{k} \|p_i\|_G \, \|X_i\|_\infty, \; A = \sum_{i=1}^{k} p_i \otimes X_i \Big\}.$$
$$\|A\|_{\mathbb{R}^N \otimes_\varepsilon \mathcal{M}_d^{sa}(\mathbb{C})} := \sup \Big\{ \langle \pi \otimes \alpha, A \rangle ; \; \|\pi\|_G^* \le 1, \; \|\alpha\|_1 \le 1 \Big\}.$$

with $\mathcal{M}_d^{sa}(\mathbb{C}) \ni \alpha \to \|\alpha\|_1 = \mathrm{Tr}\,|\alpha|$ is the Schatten 1-norm (or the nuclear norm).

6.3 Incompatibility vs Nonlocality

In the setting of XOR games where Alice's measurements are fixed, we have seen in Section 6.2 that if Alice wants to know if her measurements are local she will need to compute the norm $\|\cdot\|_G$ and if this norm is less than or equal the classical bias $\beta(G)$ then we say that her measurements are $G-$Bell-local. In order to know if her measurements are compatible she will compute the compatibility tensor norm $\|\cdot\|_c$. The problem of understanding the link between the incompatibility of quantum measurements and Bell inequality violation becomes natural, in the sense that Alice should compare the two norms. We start with a reformulation, using the language of tensor norms, of the following well-established fact: an observed *violation of the Bell inequality M* implies necessarily the *incompatibility* of Alice's measurements. Mathematically, this corresponds to the upper bounding the M-Bell-locality norm of Alice's measurements by their compatibility norm.

Theorem 6.3.1. *[2, Theorem 5.8.1] Consider a N-input, 2-output nonlocal invertible game G, corresponding to a matrix $G \in \mathcal{M}_N(\mathbb{R})$. Then, for any N-tuple of self-adjoint matrices $A = (A_1, \ldots, A_N)$, we have*

$$\|A\|_G \leq \|A\|_c \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} = \|A\|_c \, \beta(G). \tag{6.1}$$

In particular, if Alice's measurements A are G-Bell-nonlocal (in the sense of Definition 6.1.2), then they must be incompatible.

Theorem 6.3.2. *[2, Theorem 5.8.2] Consider a N-input, 2-output nonlocal game G, corresponding to an* invertible *matrix $G \in \mathcal{M}_N(\mathbb{R})$. Then, for any N-tuple of self-adjoint matrices $A = (A_1, \ldots, A_N)$, we have*

$$\|A\|_c \leq \|A\|_G \|G^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}. \tag{6.2}$$

The two theorems above show that in general, we don't have the following equality $\| \cdot \|_G = \| \cdot \|_c$. Putting together Theorems 6.3.1 and 6.3.2, we recover the main result from [9]: for $N = 2$ and the CHSH matrix

$$G_{\text{CHSH}} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

we have

$$\beta(G_{\text{CHSH}}) = 1 \qquad \text{and} \qquad (G_{\text{CHSH}})^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

It follows thus, from Eqs. (6.1) and (6.2) that

$$\| \cdot \|_c = \| \cdot \|_{G_{\text{CHSH}}} \tag{6.3}$$

Up to this point, we have seen the following two inequalities relating the G-Bell-locality norm $\| \cdot \|_G$ and the compatibility norm $\| \cdot \|_c$ of a tuple of dichotomic quantum measurements:

$$\|A\|_G \leq \|A\|_c \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \qquad \text{and} \qquad \|A\|_c \leq \|A\|_G \|G^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}.$$

In this section, we ask for which (invertible) nonlocal games G, these two inequalities, used together, allow us to conclude that $\| \cdot \|_G = \| \cdot \|_c$. Such equality would prove a strong equivalence of Bell inequality violations and incompatibility for the game G, in the spirit of [9].

First, note that, for an invertible game G and a non-zero tuple of measurements A, we have

$$\|A\|_G \leq \|A\|_c \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \leq \|A\|_G \|G^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N},$$

hence

$$\|G^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \geq 1. \tag{6.4}$$

In order to deduce that $\| \cdot \|_G = \| \cdot \|_c$, one can require

$$\beta(G) = \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} = 1 \qquad \text{and} \qquad \|G^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} = 1.$$

Up to rescaling, this is equivalent to requiring that the inequality (6.4) should be saturated. We now study the equality case in (6.4), which can be seen as an "uncertainty relation" for the nonlocal game G.

Theorem 6.3.3. *[2, Theorem 5.9.6] The only invertible nonlocal games $G \in \mathcal{M}_N(\mathbb{R})$ satisfying*

$$\|G^{-1}\|_{\ell_\infty^2 \otimes_\varepsilon \ell_\infty^2} \|G\|_{\ell_1^2 \otimes_\varepsilon \ell_1^2} = 1$$

have two questions ($N = 2$) and are variants of the CHSH game: $G = aG_{CHSH}$ for some $a \neq 0$. Hence, the CHSH game (and its permutation) is the only XOR game for which the strong equivalence between incompatibility and nonlocality holds.

Chapter 7

Conclusion

To summarise, (in)compatibility and nonlocality are two fundamental non-equivalent concepts. We have investigated in Chapter 5 the compatibility of quantum measurements where we have introduced the *compatibility dimension* as a new concept to understand the effect of the Hilbert space dimension on the compatibility of the quantum measurements. As it was described in the same chapter, to make incompatible measurements compatible, one can add noise, with several noise models being established in the literature. In an ongoing project[1], we introduce such a noise model based on an indirect measurement process. During the process, the quantum state is coupled with a probe, and the total evolution is taken as random. The measurement on the probe will induce an effective noisy POVM, where the noise parameter is completely encoded in the probe. From this thought experiment, we recover noisy effective POVMs different from those introduced in the literature; we investigate how this type of noise model affects the compatibility of the POVMs. In Chapter 6, we have developed a new framework to unify the (in)compatibility of quantum measurement and nonlocality, based on the framework of nonlocal games (XOR games) and tensor norms. Alice's measurements apparatuses are fixed, and she computes two norms describing respectively (in)compatibility of her measurements and nonlocality. By comparing the norms, the only games satisfying the equality between the two norms are (in a strong sense) the CHSH game and its permutations.

To conclude, the incompatibility of quantum measurements and quantum nonlocality are fascinating topics to understand the limitations of quantum theory. Moreover, several directions can be explored for further investigation. In this thesis we have explored the link between the incompatibility of quantum measurements and nonlocality, using nonlocal games with two players with N inputs and two outputs, with an invertible game G.

In what follows we shall give some extensions and open directions that can be addressed:

- One natural question to ask is if the invertibility of the game is a necessary condition, can we find non-invertible games that give the equivalence between incompatibility and nonlocality?
- In [9] the strong equivalence between incompatibility and nonlocality is given by the equality of $\|\cdot\|_c = \|\cdot\|_G$; to satisfy it, it suffices to saturate the inequality given by the equation (6.4). Can we relax this condition to find other matrices satisfying the inequality (6.4)?
- Can we extend the framework to nonlocal games with N inputs and M outputs? Can we define a compatibility norm for such games M-outcome measurements with $M \geq 3$?
- It was shown in [91, 92] that the largest Bell violation diverges with the number of questions. One can ask if one of the two players uses incompatible measurements how it will

[1] Not included in this thesis.

affect the Bell violation. If we use more compatible measurement, is it affected by how much one can violate some given Bell inequality?

- The authors in [46, 49] have shown if we consider games with three players, N inputs, and two outputs, have shown the maximal Bell violation diverges with the number of questions. One can ask if we fix the measurement apparatus of one of the players, how it will affect quantitatively the amount by which a Bell inequality is violated?

Part II

Presentation of the papers

Chapter 8

The compatibility dimension of quantum measurements

This chapter is a reproduction of the paper [1].

We introduce the notion of compatibility dimension for a set of quantum measurements: it is the largest dimension of a Hilbert space on which the given measurements are compatible. In the Schrödinger picture, this notion corresponds to testing compatibility with ensembles of quantum states supported on a subspace, using the incompatibility witnesses of Carmeli, Heinosaari, and Toigo. We provide several bounds for the compatibility dimension, using approximate quantum cloning or algebraic techniques inspired by quantum error correction. We analyze in detail the case of two orthonormal bases, and, in particular, that of mutually unbiased bases.

8.1 Introduction

The process of measurement in quantum mechanics has many properties differentiating it from what one encounters in classical theories. First of all, Born's rule states that the outcome of a quantum measurement is probabilistic, quantum theory predicting only the probability distribution of possible outcomes. Heisenberg's uncertainty principle gives a lower bound on the joint precision with which values can be attributed to general quantum observables. Closely related to the latter is the notion of *quantum incompatibility*: there exist quantum measurements that cannot be performed simultaneously on an unknown quantum state. Incompatibility of quantum measurements has received a lot of attention from both theorists (as a signature of quantumness) and experimentalists (mainly due to the relation to Bell non-locality [9, 93, 94]).

For a pair of incompatible quantum measurements, it is well known that adding enough noise renders them compatible [95, 96]. This has been a very fruitful direction of research, see the recent review [97] and the connection to free spectrahedra [80, 81]. In this work, we study a different approach to the same problem of making measurements compatible, by *dimension reduction*. This can be understood in two equivalent ways:

- taking corners of the POVM elements (Heisenberg picture)
- restricting the sets of quantum states to a subspace (Schrödinger picture).

We introduce a measure of incompatibility of measurements from this perspective: the *compatibility dimension* of a tuple of POVMs $A^{(1)}, \ldots, A^{(g)}$ is the largest Hilbert space dimension r for which there *exists* an isometry $V : \mathbb{C}^r \to \mathbb{C}^d$ such that the reduced POVMs $V^* A^{(1)} V, \ldots, V^* A^{(g)} V$ are compatible, see Definition 8.4.4. Similarly, we define the *strong compatibility dimension* of a tuple of measurements as the largest dimension r for which *all* isometries $V : \mathbb{C}^r \to \mathbb{C}^d$ reduce the POVMs to a compatible tuple.

We study different examples and fundamental properties of these newly defined quantities. Using analytic and algebraic techniques, we prove several bounds in the most relevant cases. For the case of two von Neumann measurements, we relate the compatibility dimension to a geometric quantity encoding the relative position of the vectors of the two bases. For two noisy mutually unbiased bases, we show that, for some particular values of the noise parameters, dimensionality reduction renders incompatible measurements compatible. To do so, we prove along the way a generalization of a compatibility criterion [98] coming from quantum cloning. We relate these dimensions to the notion of incompatibility witnesses introduced in [85, 99], using the measurement / state duality. We use algebraic techniques inspired from the theory of quantum error correction to prove very general lower bounds on the compatibility dimension. Finally, we consider spin systems coming from Clifford algebras as an illuminating example.

The newly introduced measure, the *compatibility dimension* of a tuple of quantum measurements, sheds light on the complex phenomenon of quantum incompatibility. It is a discrete measure of incompatibility: compatible POVMs have maximal compatibility dimension (equal to that of the ambient Hilbert space), while smaller compatibility dimensions indicate a higher robustness of incompatibility. We provide a plethora of results regarding this measure, of both analytical and algebraic flavor, focusing on important classes of POVMs, such as noisy mutually unbiased von Neumann measurements. We leave a certain number of questions regarding the compatibility dimension open, and hope that our work will stimulate further research in this direction.

Our paper is organized as follows. In Section 9.3 we recall the main definitions and the basic properties of quantum measurements, focusing on the notion of compatibility. We present in Section 8.3 a generalization of a compatibility criterion using asymmetric cloning. Section 8.4 contains the main definitions of the paper, that of the (strong) compatibility dimension. We switch to the Schrödinger picture in Section 8.5, relating the compatibility dimension to incompatibility witnesses and discrimination of state super-ensembles. Sections 8.6 and 8.7 are devoted to two important examples: von Neumann measurements and (noisy) mutually unbiased bases. In Section 8.8 we use techniques inspired by quantum error correction to provide very general lower bounds for the compatibility dimension. Finally, we study spin systems in Section 8.9, obtaining lower bounds for the strong compatibility dimension. We conclude with a list of open questions and directions for further research.

8.2 Compatibility of quantum measurements

We gather in this section the main definitions and basic facts from the theory of quantum measurements. In quantum mechanics, to quantum systems we associate a complex Hilbert space $\mathcal{H}$. In this paper, we shall focus on finite dimensional Hilbert spaces, so we shall write $\mathcal{H} \cong \mathbb{C}^d$ for a positive integer d, the number of degrees of freedom of the quantum system. We denote by $\mathcal{M}_d$ the vector space of $d \times d$ complex matrices. The *states* of a quantum system are mathematically modelled by *density matrices*

$$\mathcal{M}_d^{1,+} := \{\rho \in \mathcal{M}_d \,:\, \rho \geq 0 \text{ and } \operatorname{Tr} \rho = 1\},$$

where $\rho \geq 0$ means that the matrix ρ is positive semidefinite (i.e. ρ is self-adjoint and has non-negative eigenvalues).

The measurement process is modelled in quantum mechanics by *observables*. This formalism allows to obtain the probability distribution of the possible outcomes, as well as the state of the system after the measurement (the *wave function collapse*). In this work, we are interested in the probabilities of outcomes only, so we shall use the framework of POVMs. We write $[n] := \{1, 2, \ldots, n\}$.

Definition 8.2.1. *A* positive operator valued measure *(POVM) on $\mathcal{M}_d$ is a tuple $A = (A_1, \ldots, A_k)$*

of self-adjoint operators from $\mathcal{M}_d$ which are positive semidefinite and sum up to the identity:

$$\forall i \in [k], \quad A_i \geq 0 \qquad \text{and} \qquad \sum_{i=1}^{k} A_i = I_d.$$

When measuring a POVM A on a quantum system in state ρ, we obtain a random outcome

$$\forall i \in [k], \qquad \mathbb{P}(\text{outcome} = i) = \mathrm{Tr}[\rho A_i].$$

The properties of the POVM operators A_i (called *quantum effects*) ensure that the vector $(\mathrm{Tr}[\rho A_i])_{i=1}^k$ is a probability vector. Note that this mathematical formalism does not account for what happens with the quantum particle after the measurement; we say that the particle is destroyed in the process of measurement, see Figure 9.2.

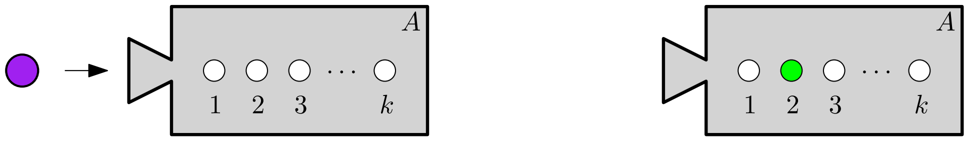


Figure 8.1: Diagrammatic representation of a quantum measurement. Left: a quantum particle enters a measurement apparatus. Right: after the measurement is performed, the particle is destroyed, and the apparatus displays the classical outcome (here, 2).

An important class of POVMs are *von Neumann measurements*, where $A_i = |a_i\rangle\langle a_i|$, $i \in [d]$, for an orthonormal basis $\{|a_i\rangle\}_{i=1}^d$ of $\mathbb{C}^d$. On the other side of the spectrum, there are *trivial* POVMs, where $B_j = q_j I_d$, for some probability vector $q = (q_1, \ldots, q_k)$. Note that for trivial POVMs, the outcome probabilities are given by the vector q, independently of the quantum state ρ that is being measured. The special case of equi-probability $q_j = 1/k$ will be of interest in this paper: we define the notion of noisy POVMs, with respect to the *random* or *uniform* noise model (see [97]).

Definition 8.2.2. *For a POVM A and a parameter $t \in [0,1]$, we define the* noisy version $\mathcal{N}_t[A]$ *of A by*

$$\mathcal{N}_t[A]_i = tA_i + (1-t)\frac{I_d}{k},$$

where k is the number of outcomes of A. In other words, $\mathcal{N}_t[A]$ is the convex combination, with weight t, between A and the uniform trivial POVM $(I_d/k, \ldots, I_d/k)$.

Similarly, for g-tuples of POVMs $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$, we define

$$\mathcal{N}_{\mathbf{t}}[\mathbf{A}] = (\mathcal{N}_{t_1}[A^{(1)}], \ldots, \mathcal{N}_{t_g}[A^{(g)}]),$$

for a vector $\mathbf{t} \in [0,1]^g$. If the vector $\mathbf{t}$ is constant, $\mathbf{t} = (t, t, \ldots, t)$, we write $\mathcal{N}_t[\mathbf{A}] := \mathcal{N}_{\mathbf{t}}[\mathbf{A}]$.

Note that in the definition above, we allow POVMs having possibly different number of outcomes.

Of central importance in this work will be the following notion.

Definition 8.2.3. *Given an isometry $V : \mathbb{C}^r \to \mathbb{C}^d$ and a POVM $A = (A_1, \ldots, A_k)$ on $\mathcal{M}_d$, we define the* reduced POVM *on $\mathcal{M}_r$*

$$V^*AV := (V^*A_1V, \ldots, V^*A_kV).$$

We record here the following result, which will be used later in the paper.

Lemma 8.2.4. *For a POVM A on $\mathcal{M}_d$ and an isometry $V : \mathbb{C}^r \to \mathbb{C}^d$, we have*

$$V^* \mathcal{N}_t[A] V = \mathcal{N}_t[V^* A V].$$

Proof. This simple fact follows from the special type of noise we use:

$$V^* \mathcal{N}_t[A]_i V = t V^* A_i V + (1-t)\frac{V^* I_d V}{k} = t V^* A_i V + (1-t)\frac{I_r}{k} = \mathcal{N}_t[V^* A V]_i.$$

□

We introduce now the notion of *compatibility* for POVMs, which is central to this paper. Physically, this notion is motivated by the following scenario. Suppose we want to measure two different physical quantities (modelled by two POVMs A and B) on a given quantum particle in a state ρ. Since the particle is destroyed after performing a given measurement, we cannot measure simultaneously A and B. However, measuring A and B on ρ can be simulated by measuring a different POVM C, and then *classically* post-processing the output of C to a pair of outcomes (i, j) for A, respectively B, see Figure 9.3. Famously, there are pairs of POVMs A and B for which there is no such C, like the position and momentum operators of a particle in one dimension: it is impossible to attribute an exact value to both position and momentum observables at the same time.

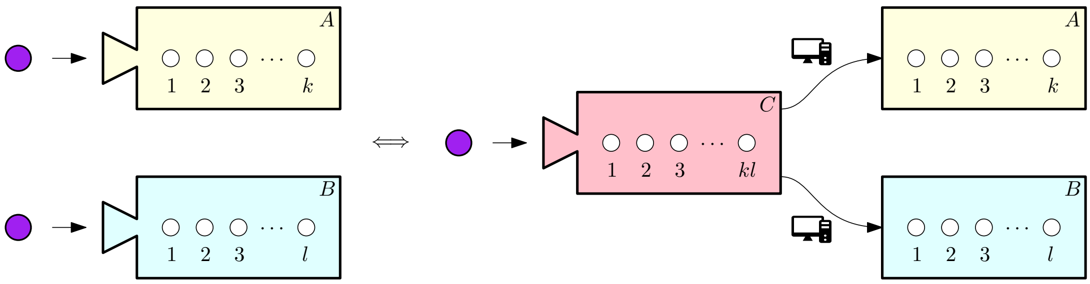


Figure 8.2: The simultaneous measurement of A and B is simulated by the measurement of C on a single copy of the quantum particle, followed by a classical post-processing of the output of C.

Mathematically, we have the following important definition, see, e.g., the excellent review paper [66].

Definition 8.2.5. *Two POVMs $A = (A_1, \ldots, A_k)$, $B = (B_1, \ldots, B_l)$ on $\mathcal{M}_d$ are called* compatible *if there exists a POVM $C = (C_{11}, \ldots, C_{kl})$ on $\mathcal{M}_d$ such that A and B are its respective* marginals*:*

$$\forall i \in [k], \qquad A_i = \sum_{j=1}^{l} C_{ij}$$

$$\forall j \in [l], \qquad B_j = \sum_{i=1}^{k} C_{ij}.$$

If this is the case, the POVM C is called a joint measurement *of A and B.*

More generally, a g-tuple of POVMs $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$ is called compatible if there exists a POVM C with outcome set $[k_1] \times \cdots \times [k_g]$ such that, for all $x \in [g]$, the POVM $A^{(x)}$ is the *x-th marginal of C:*

$$\forall i_x \in [k_x], \qquad A^{(x)}_{i_x} = \sum_{i_1=1}^{k_1} \cdots \sum_{i_{x-1}=1}^{k_{x-1}} \sum_{i_{x+1}=1}^{k_{x+1}} \cdots \sum_{i_g=1}^{k_g} C_{i_1 i_2 \cdots i_g}$$
$$= \sum_{\substack{\mathbf{j} \in [k_1]\times\cdots\times[k_g] \\ j_x = i_x}} C_{\mathbf{j}}.$$

There is a lot of literature about the compatibility relation for quantum measurements, see [66]. Let us just mention here that in the case of two POVMs A, B where at least one of them is projective (i.e. the effect operators are projections), compatibility is equivalent to commutativity $[A_i, B_j] = 0$, for all $(i, j) \in [k] \times [l]$, see [100, Proposition 8].

Given a pair of incompatible POVMs A and B, it is always possible to render them compatible by mixing in some noise:

$$\forall A, B \text{ POVMs}, \quad \mathcal{N}_{1/2}[A] \text{ and } \mathcal{N}_{1/2}[B] \text{ are compatible.}$$

Whether smaller amounts of noise suffice to render arbitrary POVMs compatible [96] is a very important ongoing research question, see [97] for a recent review, and [80, 81] for a novel approach based on free spectrahedra. In this work, we introduce and study a different method of achieving compatibility of POVMs: instead of mixing in noise, we *reduce their dimension.*

8.3 Compatibility criteria from asymmetric cloning

We present now a generalization of the compatibility criterion from [98] to the case of several POVMs and asymmetric noise parameters. We obtain a necessary condition for the compatibility of a tuple of POVMs, which is in a sense dual to the asymmetric cloning problem.

First, let us recall some basic facts about (asymmetric) cloning. It was shown that in quantum mechanics we cannot make exact copies of an arbitrary unknown quantum state [23]. This fact was formulated as *the no-cloning theorem*, which is one of the fundamental differences between the classical and the quantum worlds. To precisely state a quantitative version of this fundamental fact, let us recall the basic definitions of completely positive maps and quantum channels; we refer the reader interested in background material on quantum information theory to the monograph [14].

Definition 8.3.1. *A linear map $\Phi : \mathcal{M}_d \to \mathcal{M}_D$ is called* completely positive *if for all $K \geq 1$ and $X \in \mathcal{M}_d \otimes \mathcal{M}_K$, we have*

$$X \geq 0 \implies [\Phi \otimes \mathrm{id}_K](X) \geq 0,$$

where id_K denotes the identity map. If, moreover, the map Φ is trace preserving

$$\forall Y \in \mathcal{M}_d, \qquad \operatorname{Tr} \Phi(Y) = \operatorname{Tr} Y,$$

then Φ is called a quantum channel.

The no-cloning theorem can be precisely formulated as follows: for any number of clones $g \geq 2$, there is no quantum channel $\Phi : \mathcal{M}_d \to \mathcal{M}_d^{\otimes g}$ with the property that

$$\forall \rho \in \mathcal{M}_d^{1,+}, \forall j \in [g], \qquad \operatorname{Tr}_{[g]\setminus\{j\}} \Phi(\rho) = \rho.$$

The relation above means that there is no universal $1 \to g$ quantum cloner such that the j-th marginal of the output is equal to the input, for all $j \in [g]$.

The *asymmetric quantum approximate cloning* problem asks whether a quantum channel exists which *approximately* clones any input state. The degree of approximation can vary with the index of the marginal (i.e. clone) in the asymmetric setting. Symmetric approximate cloning was completely described in [101, 102] (using different figures of merit for the quality of the clones), while the asymmetric case was studied in [103, 104]. Physically, approximate cloning can be seen as a way to go around the obstruction from the no-cloning theorem by adding noise: our goal is to produce imperfect, noisy copies of the original input state. We formalize the above in the following definition (see also [80]).

Definition 8.3.2. *The approximation parameters of physical $1 \to g$ asymmetric cloners on $\mathbb{C}^d$ are described by the following set:*

$$\Gamma^{clone}(g,d) := \Big\{ \mathbf{s} \in [0,1]^g \;:\; \exists \Phi : \mathcal{M}_d \to \mathcal{M}_d^{\otimes g} \text{ quantum channel such that } \\ \forall \rho \in \mathcal{M}_d, \forall j \in [g], \quad \operatorname{Tr}_{[g]\setminus\{j\}} \Phi(\rho) = s_j \rho + (1-s_j)\frac{I_d}{d} \Big\}.$$

The classical no-cloning theorem states that perfect clones are impossible: for all $g, d \geq 2$, $(1,1,\ldots,1) \notin \Gamma^{clone}(g,d)$. In [104], the *optimal asymmetric cloning parameters* were computed explicitly (see also [103] for an alternative approach, based on representation theory). Those results, stated in term of fidelities, can be restated in our language of depolarizing channels using [81, Proposition 6.5], which uses the *twirling* operation to symmetrize the marginals of an optimal cloner.

Theorem 8.3.3. *[104, Section 2.3, Theorem 1] For all $g, d \geq 2$, the* optimal *asymmetric cloning parameters are given by*

$$\begin{aligned} \partial\Gamma^{clone}(g,d) &= \{ \mathbf{s} \in (0,1]^g \;:\; \forall \varepsilon > 0,\, (1+\varepsilon)s \notin \Gamma^{clone}(g,d) \} \\ &= \Bigg\{ \mathbf{s} \in (0,1]^g \;:\; (g+d-1)\left[g - d^2 + d + (d^2-1)\sum_{i=1}^{g} s_i \right] = \\ &\qquad \left(\sum_{i=1}^{g} \sqrt{s_i(d^2-1)+1} \right)^2 \Bigg\}. \end{aligned}$$

The task of cloning quantum states can be reinterpreted in the Heisenberg picture of quantum mechanics by looking at the dual map of a channel; this operation acts naturally on quantum measurements. In this picture, the dual property of producing imperfect clones is having noisy measurements. Let us define the asymmetric dual map for the POVMs, and the corresponding set of cloning parameters. Consider the set of parameters for this dual maps:

$$\tilde{\Gamma}^{clone}(g,d) := \Big\{ \mathbf{s} \in [0,1]^g \;:\; \exists \Psi : \mathcal{M}_d^{\otimes g} \to \mathcal{M}_d \text{ unital and completely positive such that } \\ \forall X \in \mathcal{M}_d,\, \forall j \in [g], \quad \Psi(I^{\otimes(j-1)} \otimes X \otimes I^{\otimes(g-j)}) = s_j X + (1-s_j)\frac{\operatorname{Tr} X}{d} I \Big\}. \tag{8.1}$$

Proposition 8.3.4. *The dual and the primal sets of cloning parameters are identical: $\forall g, d \geq 2$,*

$$\tilde{\Gamma}^{clone}(g,d) = \Gamma^{clone}(g,d).$$

Proof. Let us prove the first inclusion $\tilde{\Gamma}^{clone}(g,d) \subseteq \Gamma^{clone}(g,d)$, the other one being similar. Let $\mathbf{s} \in \tilde{\Gamma}^{clone}(g,d)$, and consider the unital completely positive map $\Psi : \mathcal{M}_d^{\otimes g} \to \mathcal{M}_d$ having the tuple $\mathbf{s}$ as an approximation parameter. Let us define $\Phi := \Psi^*$; since Ψ is unital and completely positive, Φ is a quantum channel [14, Section 2.2]. For any quantum state $\rho \in \mathcal{M}_d^{1,+}$, any matrix

$X \in \mathcal{M}_d$, and any $j \in [g]$, we have

$$\begin{aligned}
\operatorname{Tr}\left[\left(\operatorname{Tr}_{[g]\setminus\{j\}} \Phi(\rho)\right) \cdot X\right] &= \operatorname{Tr}\left[\Phi(\rho) \cdot \left(I_d^{\otimes(j-1)} \otimes X \otimes I_d^{\otimes(g-j)}\right)\right] \\
&= \operatorname{Tr}\left[\rho \cdot \Psi\left(I_d^{\otimes(j-1)} \otimes X \otimes I_d^{\otimes(g-j)}\right)\right] \\
&= \operatorname{Tr}\left[\rho \cdot \left(s_j X + (1-s_j)\frac{\operatorname{Tr} X}{d} I_d\right)\right] \\
&= s_j \operatorname{Tr}[\rho X] + (1-s_j)\frac{\operatorname{Tr} X}{d} \\
&= \operatorname{Tr}\left[\left(s_j \rho + (1-s_j)\frac{I_d}{d}\right) \cdot X\right],
\end{aligned}$$

proving that, for all ρ and j, $\operatorname{Tr}_{[g]\setminus\{j\}} \Phi(\rho) = s_j\rho + (1-s_j)\frac{I_d}{d}$. Hence, $\Phi = \Psi^*$ is a valid quantum cloner with parameter $\mathbf{s}$, which finishes the proof. □

We shall now use the above results on quantum cloning to generalize the following compatibility criterion. We denote by $\lambda_{\min}(X)$ the minimal eigenvalue of a self-adjoint operator X.

Proposition 8.3.5. *[79, Proposition III.3] Consider two POVMs A and B on $\mathcal{M}_d$ satisfying*

$$\begin{aligned}
\lambda_{\min}(A_i) &\geq \frac{1}{2(d+1)} \operatorname{Tr} A_i \qquad \forall i \\
\lambda_{\min}(B_j) &\geq \frac{1}{2(d+1)} \operatorname{Tr} B_j \qquad \forall j.
\end{aligned}$$

Then, A and B are compatible.

We provide next a generalization of the compatibility criterion above for g-tuples of POVMs and asymmetric noise parameters.

Theorem 8.3.6. *Let $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$ be a g-tuple of POVMs on $\mathcal{M}_d$ having, respectively, $k_1, \ldots, k_g$ outcomes. Define, for all $x \in [g]$,*

$$s_x := 1 - \min_{i \in [k_x]} \frac{d\lambda_{\min}(A_i^{(x)})}{\operatorname{Tr} A_i^{(x)}} \in [0,1].$$

If $\mathbf{s} \in \Gamma^{clone}(g,d)$, then the POVMs in $\mathbf{A}$ are compatible.

Proof. Note first that the assumptions in the statement are equivalent to the following set of inequalities:

$$\forall x \in [g], \forall i \in [k_x], \qquad \lambda_{\min}(A_i^{(x)}) \geq \frac{1-s_x}{d} \operatorname{Tr} A_i^{(x)}. \tag{8.2}$$

Let Ψ be the unital completely positive map appearing in the definition of $\tilde{\Gamma}^{clone}(g,d) \ni \mathbf{s}$. Let us define, for all $x \in [g]$ such that $s_x > 0$,

$$B_i^{(x)} := \frac{1}{s_x}\left(A_i^{(x)} - (1-s_x)\frac{\operatorname{Tr} A_i^{(x)}}{d} I_d\right), \quad \forall i \in [k_x].$$

If $s_x = 0$, put $B_i^{(x)} = I_d/k_x$ for all $i \in [k_x]$. We claim that $\mathbf{B} = (B^{(x)})_{x\in[g]}$ form a tuple of POVMs on $\mathcal{M}_d$. Indeed, it is easy to see that $B^{(x)}$ is normalized for all x, and that the positivity of $B_i^{(x)}$ follows from Eq. (8.2) for all i. Moreover, we have $\operatorname{Tr} B_i^{(x)} = \operatorname{Tr} A_i^{(x)}$ for all x, i.

Define, for $\mathbf{i} = (i_1, \ldots, i_g) \in [k_1] \times \cdots \times [k_g]$,

$$C_{\mathbf{i}} := \Psi(B^{(1)}_{i_1} \otimes \cdots \otimes B^{(g)}_{i_g}).$$

Since Ψ is (completely) positive and unital, it follows that C is a POVM on $\mathcal{M}_d$ with $k_1 \cdots k_g$ outcomes. From (8.1), it follows that the x-marginal of C is given by

$$\forall i_x \in [k_x], \qquad \sum_{i_1,\ldots,i_{x-1},i_{x+1},\ldots,i_g} C_{\mathbf{i}} = \Psi\left(I_d^{\otimes(x-1)} \otimes B^{(x)}_{i_x} \otimes I_d^{\otimes(g-x)}\right) = s_x B^{(x)}_{i_x} + (1-s_x)\frac{\operatorname{Tr} B^{(x)}_{i_x}}{d} I_d = A^{(x)}_{i_x},$$

showing that the POVMs $\mathbf{A}$ are compatible, with joint measurement C. □

Note that Proposition 8.3.5 follows from Theorem 8.3.6 using the fact that

$$\left(\frac{d+2}{2(d+1)}, \frac{d+2}{2(d+1)}\right) \in \Gamma^{clone}(2, d)$$

for all $d \geq 2$.

8.4 Compatibility dimensions — definition and examples

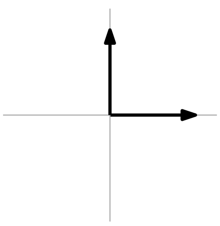
This section contains the definition of the main objects we study in the paper: the different notions of *compatibility dimension*.

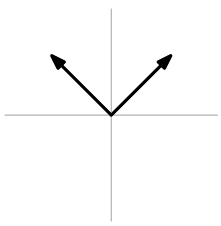
We start with an example in order to provide some intuition about dimension reduction. Consider $A = \{|i\rangle\langle i|\}_{i=1}^5$ the von Neumann measurement in the computational basis of $\mathbb{C}^5$, and the POVM $B = (B_i)_{i=1}^5$ given by

$$B_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_3 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_4 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that we have $A_5 = B_5 = |5\rangle\langle 5|$. On the two-dimensional space spanned by $|1\rangle$, $|2\rangle$ (resp. $|3\rangle$, $|4\rangle$), the operators $A_{1,2}$ and $B_{1,2}$ (resp. $A_{3,4}$ and $B_{3,4}$) perform the von Neumann measurements in the two bases below (left basis for A and right basis for B):





Since the projective measurements A, B do not correspond to the same orthonormal basis, they are not compatible. However, one can render them compatible by considering their *reduction* (see Definition 8.2.3) on a three-dimensional space. Indeed, consider the isometry $V : \mathbb{C}^3 \to \mathbb{C}^5$ given by

$$V = |1\rangle\langle 1| + |3\rangle\langle 2| + |5\rangle\langle 3|. \tag{8.3}$$

We have

$$V^*AV = (|1\rangle\langle 1|, 0, |2\rangle\langle 2|, 0, |3\rangle\langle 3|)$$

while

$$V^*BV = \left(\frac{|1\rangle\langle 1|}{2}, \frac{|1\rangle\langle 1|}{2}, \frac{|2\rangle\langle 2|}{2}, \frac{|2\rangle\langle 2|}{2}, |3\rangle\langle 3|\right).$$

Hence, although the original POVMs A, B were incompatible, their reduced versions V^*AV and V^*BV are commuting, hence compatible. From a physical perspective, we have found a 3-dimensional subspace $E = \mathrm{Ran}(V) \subseteq \mathbb{C}^5$ such that the POVMs A, B look compatible when measuring quantum states supported on E. This connection with quantum states shall be discussed in details in Section 8.5.

We now introduce the main quantities of interest in this work, starting with the most general one. We recall that, in the theory of partial ordered sets, a *down-set* is a set X with the property that if $x \in X$ and $y \preceq x$, then $y \in X$ ("$\preceq$" denotes the partial order relation).

Definition 8.4.1. *Given a g-tuple of POVMs $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$, define their* compatibility down-set *as*

$$\mathcal{C}(\mathbf{A}) := \{E \subseteq \mathbb{C}^d \,|\, V^*\mathbf{A}V \text{ are compatible for some isometry } V \text{ with } \mathrm{Ran}(V) = E\}. \tag{8.4}$$

In other words, the compatibility down-set is the set of subspaces on which the POVMs $\mathbf{A}$ are compatible.

We gather some basic facts about the sets $\mathcal{C}(\mathbf{A})$ in the following proposition. We denote by $\mathcal{S}_r(\mathbb{C}^d)$ the Grassmannian of all r-dimensional subspaces of $\mathbb{C}^d$

$$\mathcal{S}_r(\mathbb{C}^d) := \{E \subseteq \mathbb{C}^d \mid \dim E = r\}$$

and we also write

$$\mathcal{S}(\mathbb{C}^d) = \bigsqcup_{r=0}^{d} \mathcal{S}_r(\mathbb{C}^d)$$

for the full Grassmannian.

Proposition 8.4.2. *The set $\mathcal{C}(\mathbf{A})$ has the following properties:*

- *$\mathcal{C}(\mathbf{A})$ is a down-set in the modular lattice $\mathcal{S}(\mathbb{C}^d)$ of subspaces of $\mathbb{C}^d$*
- *$\mathcal{C}(\mathbf{A})$ contains all the 1-dimensional subspaces*
- *the POVMs $\mathbf{A}$ are compatible if and only if $\mathcal{C}(\mathbf{A}) = \mathcal{S}(\mathbb{C}^d)$*
- *$\mathcal{C}(\mathbf{A})$ is graded by $r = \dim E$:*

$$\mathcal{C}(\mathbf{A}) = \bigsqcup_{r=0}^{d} \mathcal{C}_r(\mathbf{A}),$$

where

$$\mathcal{C}_r(\mathbf{A}) := \mathcal{C}(\mathbf{A}) \cap \mathcal{S}_r(\mathbb{C}^d).$$

- *in Definition 8.4.1, the words "some isometry" can be replaced by "all isometries".*

Proof. Let us prove the first claim. Consider a subspace $F \subseteq E$ of dimension $\dim F = s$ and choose an isometry $W : \mathbb{C}^s \to \mathbb{C}^d$ such that $\mathrm{Ran}\, W = F$. Since $F \subseteq E$, we have $W = VV^*W$. We have thus $W^*\mathbf{A}W = W^*V(V^*\mathbf{A}V)V^*W$. The compatibility of $W^*\mathbf{A}W$ follows then from that of $V^*\mathbf{A}V$.

The fact that $\mathcal{C}(\mathbf{A})$ contains all vector lines follows from commutativity. Having $\mathbb{C}^d \in \mathcal{C}(\mathbf{A})$ is clearly equivalent to the compatibility of the POVMs in $\mathbf{A}$.

The final claim follows from the observation that any two isometries $V_{1,2} : \mathbb{C}^r \to \mathbb{C}^d$ with $\mathrm{Ran}\, V_{1,2} = E$ are related via a unitary $U : \mathbb{C}^r \to \mathbb{C}^r$ by $V_2 = V_1 U$, and from the fact that conjugation by a global unitary does not change compatibility. $\square$

Remark 8.4.3. *The map $\mathbf{A} \mapsto \mathcal{C}(\mathbf{A})$ is an anti-order-morphism with respect to the pre- and post-processing order relations on the set of tuples of POVMs, see [68, Section 5].*

Since the lattice of subspaces of $\mathbb{C}^d$ is a cumbersome object to work with, we consider a coarse-grained version of Definition 8.4.1, where we keep track only of the dimension of the subspaces.

Definition 8.4.4. *Given a g-tuple of POVMs $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$ on a d-dimensional quantum system, we define their* compatibility dimension *as the largest dimension r for which there* exists *an isometry $V : \mathbb{C}^r \to \mathbb{C}^d$ reducing the POVMs to a compatible g-tuple:*

$$\begin{aligned} R(\mathbf{A}) &:= \max\{r \in [d] \, : \, \exists V : \mathbb{C}^r \to \mathbb{C}^d \text{ isom. s.t. } V^* A^{(1)} V, \ldots, V^* A^{(g)} V \text{ are comp.}\} \\ &= \max\{r \in [d] \, : \mathcal{C}_r(\mathbf{A}) \neq \emptyset\}. \end{aligned} \tag{8.5}$$

Similarly, we define the strong compatibility dimension *of a g-tuple of POVMs $\mathbf{A}$ as the largest dimension r for which* all *isometries $V : \mathbb{C}^r \to \mathbb{C}^d$ reduce the POVMs to a compatible g-tuple:*

$$\begin{aligned} \bar{R}(\mathbf{A}) &:= \max\{r \in [d] \, : \, \forall V : \mathbb{C}^r \to \mathbb{C}^d \text{ isom., } V^* A^{(1)} V, \ldots, V^* A^{(g)} V \text{ are comp.}\} \\ &= \max\{r \in [d] \, : \mathcal{C}_r(\mathbf{A}) = \mathcal{S}_r(\mathbb{C}^d)\}. \end{aligned} \tag{8.6}$$

We have the following simple observations, which follow directly from the definition.

Remark 8.4.5. *For all g-tuples $\mathbf{A}$ of POVMs on $\mathcal{M}_d$, we have*

$$1 \leq \bar{R}(\mathbf{A}) \leq R(\mathbf{A}) \leq d.$$

We also have $\bar{R}(\mathbf{A}) = d \iff R(A) = d \iff A^{(1)}, \ldots, A^{(g)}$ are compatible quantum measurements.

For the example of the two POVMs A, B introduced at the beginning of this section, using the isometry V from (8.3), we have $R(A, B) \geq 3$. On the other hand, using the isometry

$$W = |1\rangle\langle 1| + |2\rangle\langle 2| + |5\rangle\langle 3|,$$

we have $W^*AW = (|1\rangle\langle 1|, |2\rangle\langle 2|, 0, 0, |3\rangle\langle 3|)$, while

$$W^*BW = \left(\frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, 0, 0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

Note that the two POVMs W^*AW, W^*BW are incompatible, proving that $\bar{R}(A, B) \leq 2$; we have thus provided an example where $\bar{R} < R$.

In this work, we shall focus mostly on the quantity R. Let us point out however that the measure $\bar{R}$ has been related in [80, 81] to the inclusion problem for different levels of the matrix diamond and its generalizations into a free spectrahedron defined by $\mathbf{A}$; we shall not pursue these aspects in this work.

8.5 Restricted incompatibility witnesses

We provide in this section a characterization of the incompatibility dimension with the help of incompatibility witnesses. This point of view is "dual" in some sense to the original definition from Section 8.4, providing an operational interpretation of the dimensions $R(\mathbf{A})$ and $\bar{R}(\mathbf{A})$ as the size of the support of superensembles of quantum states allowing for an advantage in a state discrimination protocol (see Theorem 8.5.4).

Several notions of incompatibility witnesses have been considered in the literature, by [105], [85], and [81]. We shall consider here the second listed approach, developed in [85, 99], which has a very nice operational interpretation, in terms of state ensembles distinguishability, with prior vs. posterior information. The same connection between incompatibility witnesses and state ensemble distinguishability was discovered independently in [106–108].

Let us first describe the state discrimination protocols which provide the framework for incompatibility witnesses, following [99]. Recall that a state ensemble $\mathcal{E}$ is a set of quantum states $\sigma_1, \ldots, \sigma_k \in \mathcal{M}_d^{1,+}(\mathbb{C})$, together with a probability vector $p = (p_1, \ldots, p_k)$. We also consider *superensembles* E, which are g-tuples of state ensembles $(\mathcal{E}^{(1)}, \ldots, \mathcal{E}^{(g)})$, together with a probability measure $q = (q_1, \ldots, q_g)$. Note that we do not require that the number of elements in each ensemble (respectively $k_1, \ldots, k_g$) is identical. We consider now two superensemble discrimination protocols, which differ only in the timing when the state ensemble label is communicated. The main idea of the protocol is presented in Figure 8.3, while the details of the explicit steps of the protocol are given in Table 8.1.

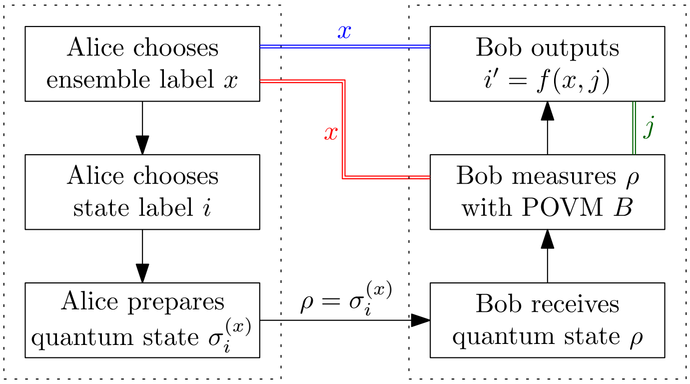


Figure 8.3: The superensemble discrimination protocol, with its two variants: prior information and posterior information.

The input of the protocol is a superensemble E, and we shall be interested in the success probability $\mathbb{P}_{guess}$, of Bob correctly identifying to which ensemble element Alice's state corresponds to. In other words, we are interested in Bob's best choice of a POVM B such that the probability that the protocol succeeds (i.e. $i = i'$) is maximal. Let us consider the two scenarios separately. In the scenario with prior information, Bob knows from which ensemble $\mathcal{E}^{(x)}$ the state ρ has been sampled, so he can choose B to be the POVM which discriminates best the (weighted) states from $\mathcal{E}^{(x)}$. We obtain

$$\mathbb{P}_{guess}^{prior}(E) = \sup \left\{ \sum_{x=1}^{g} q_x \langle \mathcal{E}^{(x)}, B^{(x)} \rangle \; : \; B^{(1)}, \ldots, B^{(g)} \text{ POVMs} \right\},$$

where we use $\langle \cdot, \cdot \rangle$ to denote the state ensemble-POVM duality:

$$\langle \mathcal{E}^{(x)}, B^{(x)} \rangle := \sum_{i=1}^{k_x} p_i^{(x)} \operatorname{Tr}[\sigma_i^{(x)} B_i^{(x)}].$$

In the scenario with posterior information, Bob does not have the knowledge of x at the time he performs the quantum measurement, and it has been shown in [99, Eq. (13)] that

$$\mathbb{P}_{guess}^{post}(E) = \sup \left\{ \sum_{x=1}^{g} q_x \langle \mathcal{E}^{(x)}, C^{(x)} \rangle \; : \; C^{(1)}, \ldots, C^{(g)} \text{ compatible POVMs} \right\},$$

The formula above can be understood as follows: since at the time he performs the measurement, Bob does not know from which ensemble $\mathcal{E}^{(x)}$ the state ρ is sampled from, his best bet is to perform a measurement with a large outcome set and then, once he learns the ensemble label x, to perform a classical post-processing of his measurement outcome j and the ensemble label x. This classical post-processing is equivalent to Bob measuring a joint POVM C of *compatible* POVMs $C^{(1)}, \ldots, C^{(g)}$, having respectively $k_1, \ldots, k_g$ outcomes, see [99, Proposition 1]. Since the set over which the supremum is considered is smaller in this scenario, we have $\mathbb{P}_{guess}^{prior}(E) \geq \mathbb{P}_{guess}^{post}(E)$.

Step	Prior information	Posterior information
1	Alice chooses randomly an ensemble label $x \in [g]$, using probabilities q	
2	Alice chooses randomly a state label $i \in [k_x]$, using probabilities $p^{(x)}$	
3	Alice sends the quantum state $\rho = \sigma_i^{(x)}$ to Bob	
4	Alice sends the ensemble label x to Bob	
5	Bob receives the (unknown) quantum state ρ	
6	Bob chooses a POVM B and measures ρ, obtaining an output j	
7		Alice sends the ensemble label x to Bob
8	Bob outputs $i' = f(x, j)$	
9	The protocol succeeds if $i' = i$	

Table 8.1: Superensemble discrimination protocols, with prior and posterior information. In the prior information scenario, Alice sends Bob the ensemble label x before Bob makes his measurement, allowing him to choose a POVM depending on the value x. In the posterior information scenario, Bob only learns x after performing his measurement, which cannot depend on x.

Next, Carmeli, Heinosaari and Toigo define incompatibility witnesses as follows.

Definition 8.5.1 ([85, 99]). *An* incompatibility witness *is a superensemble E such that $\mathbb{P}_{guess}^{prior}(E) > \mathbb{P}_{guess}^{post}(E)$.*

Incompatibility witnesses are used to detect incompatibility of g-tuples of POVMs in an obvious manner: given $\mathbf{A} = (A^{(1)}, \ldots A^{(g)})$, we have

$$\sum_{x=1}^{g} q_x \langle \mathcal{E}^{(x)}, A^{(x)} \rangle =: \langle E, \mathbf{A} \rangle > \mathbb{P}_{guess}^{post}(E) \implies \mathbf{A} \text{ are incompatible.} \tag{8.7}$$

Obviously, for any g-tuple of POVMs $\mathbf{A}$, we have $\langle E, \mathbf{A} \rangle \leq \mathbb{P}_{guess}^{prior}(E)$; the incompatibility witness E detect the incompatibility of $\mathbf{A}$ only when

$$\langle E, \mathbf{A} \rangle \in (\mathbb{P}_{guess}^{post}(E), \mathbb{P}_{guess}^{prior}(E)].$$

Importantly, Carmeli, Heinosaari and Toigo establish the following converse to (8.7).

Theorem 8.5.2. *[85, Theorem 2] A g-tuple* $\mathbf{A}$ *of POVMs on* $\mathcal{M}_d$ *are compatible if and only if, for all incompatibility witnesses* E *on* $\mathbb{C}^d$*, we have*

$$\langle E, \mathbf{A} \rangle \leq \mathbb{P}_{guess}^{post}(E).$$

We discuss now the relation between a restricted notion of incompatibility witnesses and the compatibility dimension we introduced in Section 8.4. We start with the following important definition.

Definition 8.5.3. *Given a subspace* $H \subseteq \mathbb{C}^d$*, we say that a quantum state* σ *is* supported *on* H *if* $\operatorname{Ran}(\sigma) \subseteq H$*. Equivalently,* σ *is supported on* H *if* $P_H \sigma P_H = \sigma$*, where* P_H *is the orthogonal projection on* H*. We say that an ensemble of quantum states* $\mathcal{E}$ *(resp. a superensemble* E*) is supported on* H *if all the states* $\sigma_i \in \mathcal{E}$ *with* $p_i > 0$ *are supported on* H*. We define the corresponding notion for superensembles in a similar manner.*

Our starting point is the following observation. Given an ensemble of quantum states supported on a subspace H and a POVM A, we have, for an isometry $V : \mathbb{C}^{\dim H} \to \mathbb{C}^d$ with $\operatorname{Ran} V = H$:

$$\begin{aligned}\langle \mathcal{E}, A \rangle &= \sum_{i=1}^k p_i \operatorname{Tr}[\sigma_i A_i] = \sum_{i=1}^k p_i \operatorname{Tr}[P_H \sigma_i P_H A_i] \\ &= \sum_{i=1}^k p_i \operatorname{Tr}[VV^* \sigma_i VV^* A_i] = \sum_{i=1}^k p_i \operatorname{Tr}[V^* \sigma_i VV^* A_i V] = \langle V^* \mathcal{E} V, V^* A V \rangle,\end{aligned}$$

On the other hand, any (compatible) g-tuple of POVMs $\mathbf{B} = (B^{(1)}, \ldots, B^{(g)})$ on $\mathcal{M}_{dimH}$ can be written as $\mathbf{B} = V^* \mathbf{A} V$ where $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$ is a (compatible) g-tuple of POVMs on $\mathcal{M}_d$. Indeed it is enough to define $A_i^{(x)} = VB_i^{(x)}V^* + \frac{I_d - VV^*}{k_k}$ for all $i \in [k_x]$, where k_x is the number of outcomes of $B^{(x)}$. This fact, together with the previous equation, immediately yields $\mathbb{P}_{guess}^{prior}(E) = \mathbb{P}_{guess}^{prior}(V^* E V)$ and $\mathbb{P}_{guess}^{post}(E) = \mathbb{P}_{guess}^{post}(V^* E V)$ for all superesnsembles E supported on H.

We have the following result, relating (super)ensembles supported on subspaces to the (strong) compatibility dimension of POVMs.

Theorem 8.5.4. *Given a g-tuple* $\mathbf{A}$ *of POVMs on* $\mathcal{M}_d$ *and an integer* $r \in [d]$*, we have* $R(\mathbf{A}) \geq r$ *if and only if there exists a subspace* $H \in \mathcal{S}_r(\mathbb{C}^d)$ *(i.e.* $H \subseteq \mathbb{C}^d$ *with* $\dim H = r$*) such that for all superensembles* E *supported on* H *we have*

$$\langle E, \mathbf{A} \rangle \leq \mathbb{P}_{guess}^{post}(E).$$

Similarly, $\bar{R}(\mathbf{A}) \geq r$ *if and only if for all superensembles* E *supported on subspaces of dimension* r*, the relation above holds.*

Proof. We shall only prove the first claim, leaving the proof of the second claim to the reader. The condition $R(\mathbf{A}) \geq r$ is equivalent to the existence of an isometry $V : \mathbb{C}^r \to \mathbb{C}^d$ such that the POVMs $V^* \mathbf{A} V$ are compatible. Let us fix such an isometry $V : \mathbb{C}^r \to \mathbb{C}^d$ with $\operatorname{Ran} V = H$ and start with the proof of the $\implies$ implication. For a superensemble E supported on H, we have

$$\langle E, \mathbf{A} \rangle = \langle V^* E V, V^* \mathbf{A} V \rangle \leq \mathbb{P}_{guess}^{post}(V^* E V) = \mathbb{P}_{guess}^{post}(E),$$

proving the claim. The reverse implication follows the same reasoning: the equation above is still true, and all superensembles E' on $\mathbb{C}^r$ can be written as $V^* E V$ for some E supported on H, namely $E = V E' V^*$. □

To summarize, we have shown in this section that the compatibility dimensions of a g-tuple of POVMs can be understood in terms of a superensemble distinguishability protocol, with states having restricted support in $\mathbb{C}^d$.

8.6 Two orthonormal bases

We consider in this section the case of two von Neumann measurements A and B corresponding to orthonormal bases in $\mathbb{C}^d$, say $\{|a_i\rangle\}_{i=1}^d$ and $\{|b_i\rangle\}_{i=1}^d$. The first observation that we can make is that we can assume, by a global unitary rotation, that one of the bases, say the first one, is the computational (canonical) basis in $\mathbb{C}^d$: $|a_i\rangle = |i\rangle$ for all $1 \leq i \leq d$. Let U be the unitary operator implementing the change of basis, such that the second basis is given by the columns of U, $\{|u_i\rangle\}_{i=1}^d$. With this notation, our task is now to compute, for some given unitary matrix $U \in \mathcal{U}_d$,

$$\mathcal{Z}(U) := R\left(\{|i\rangle\langle i|\}_{i=1}^d, \{|u_i\rangle\langle u_i|\}_{i=1}^d\right).$$

Consider now an isometry $V : \mathbb{C}^r \to \mathbb{C}^d$ and note that the operators $\tilde{A}_i = V^*|i\rangle\langle i|V$ and $\tilde{B}_i = V^*|u_i\rangle\langle u_i|V$ have rank at most one. Compatibility of unit rank POVMs is essentially the same as equality, up to permutation of effect operators and summing together collinear effects [109, 110]. We have thus the following lower bound; we conjecture that the bound is tight for generic, non-degenerate unitary matrices.

Proposition 8.6.1. *For any unitary operator $U \in \mathcal{U}_d$, we have*

$$\mathcal{Z}(U) \geq \max_{\substack{z \in \mathbb{C}^d \\ \sigma \in \mathfrak{S}_d}} \dim \ker(P_{z,\sigma} - U), \tag{8.8}$$

where $\mathfrak{S}_d$ is the symmetric group on d elements, and $P_{z,\sigma}$ is the generalized permutation matrix given by

$$P_{z,\sigma}(i,j) = z_j \delta_{i,\sigma(j)}, \qquad \forall i,j \in [d].$$

Proof. First, note that in (8.8) one can consider the adjoint of the operator $P_{z,\sigma} - U$, since for any matrix $X \in \mathcal{M}_d$, we have $\dim \ker X = \dim \ker(X^*)$. Consider a vector of scalars $z \in \mathbb{C}^d$ and a permutation $\sigma \in \mathfrak{S}_d$, and let $E = \ker[(P_{z,\sigma} - U)^*]$ having dimension $r := \dim E$. We have then, for some isometry $V : \mathbb{C}^r \to \mathbb{C}^d$ with range E,

$$(P_{z,\sigma} - U)^* V = 0_{d \times r} \implies V^*(P_{z,\sigma} - U) = 0_{r \times d}.$$

Hence, for any $j \in [d]$, we have

$$V^* |u_j\rangle = z_j V^* |\sigma(j)\rangle \implies V^*|u_j\rangle\langle u_j|V = |z_j|^2 V^*|\sigma(j)\rangle\langle\sigma(j)|V.$$

Hence, V^*AV and V^*BV are compatible POVMs, having collinear effect operators. □

We leave the question of computing $\mathcal{Z}(U)$ open in the general case. Even the bound from Eq. (8.8) seems to be hard to compute in general. A trivial lower bound is given by the largest multiplicity of the eigenvalues of U, corresponding to taking a constant vector z and fixing $\sigma = \mathrm{id}$. A natural candidate for the vector z is the diagonal of U, i.e. $z_i = u_{ii}$, a choice which has the merit that the matrices $P_{z,\mathrm{id}}$ and U have identical diagonals. Imposing the additional constraint $|z_i| = 1$ (i.e. $P_{z,\mathrm{id}}$ is unitary) amounts to choosing $z_i = \mathrm{phase}(u_{ii}) = u_{ii}/|u_{ii}|$, in the case of non-zero u_{ii}. These values are the solution of the following optimization problem:

$$\mathrm{argmin}_{z \in \mathbb{C}^d} \|P_{z,\mathrm{id}} - U\|_2^2,$$

with or without the additional constraint that $P_{z,\mathrm{id}}$ is unitary. The problem above is similar in nature to the bound from (8.8): the objective functions correspond to the matrices $P_{z,\mathrm{id}}$ and U being close to each other.

Example 8.6.2. *In the case of the Fourier operator $U = F_d$ given by $F_d(\alpha, \beta) = \omega^{\alpha\beta}$ with $\omega = \exp(2\pi\mathrm{i}/d)$, we have, with the choice $z_i = 1$ and $\sigma = \mathrm{id}$,*

$$\mathcal{Z}(F_d) \geq 1 + \lfloor d/4 \rfloor,$$

using the eigenvalue $\lambda = 1$ of F_d [111]. For example, in the case $d = 4$, a basis of the 2-dimensional eigenspace associated to the eigenvalue $\lambda = 1$ is given by the following two vectors:

$$(1, 0, 1, 0) \qquad \text{and} \qquad (2, 1, 0, 1).$$

For the general case, the problem of constructing a "simple" eigenbasis of F_d has received a lot of attention in the literature, see [112, 113].

8.7 Complementary bases

We shall consider in this section the problem of dimension reduction for the special case of two (noisy) mutually unbiased bases. Recall that a set of g orthonormal bases $\left\{\{|b_i^{(x)}\rangle\}_{i\in[d]}\right\}_{x\in[g]}$ are called *mutually unbiased* (MUB) [82, 83] if

$$\forall x \neq y \in [g], \forall i, j \in [d], \qquad |\langle b_i^{(x)}|b_j^{(y)}\rangle|^2 = \frac{1}{d}.$$

Such kind of bases are very important in quantum information theory. For example, it was shown in [114] that density matrices can be completely determined by making measurement in MUBs, and that this protocol is optimal, in the sense that the statistical error is minimized. The construction of such bases is deeply related to number theory and prime numbers which are very important for pure mathematical investigation while they have several applications in quantum information theory, quantum cryptography and entanglement, tomography, etc.; see [83].

Consider two mutually unbiased bases $\{|a_1\rangle, \ldots, |a_d\rangle\}$ and $\{|b_1\rangle, \ldots, |b_d\rangle\}$ in $\mathbb{C}^d$, for example the computational and the Fourier bases from Example 8.6.2. Let us introduce the noisy versions of the POVMs

$$\mathcal{N}_\lambda[A] = \left(\lambda|a_1\rangle\langle a_1| + (1-\lambda)\frac{I_d}{d}, \ldots, \lambda|a_d\rangle\langle a_d| + (1-\lambda)\frac{I_d}{d}\right)$$
$$\mathcal{N}_\mu[B] = \left(\mu|b_1\rangle\langle b_1| + (1-\mu)\frac{I_d}{d}, \ldots, \mu|b_d\rangle\langle b_d| + (1-\mu)\frac{I_d}{d}\right).$$

The values (λ, μ) for which the POVMs above are compatible have been computed in [84, 85]: for $(\lambda, \mu) \in [0, 1]^2$, $\mathcal{N}_\lambda[A]$ and $\mathcal{N}_\mu[B]$ are compatible iff

$$\lambda + \mu \leq 1 \text{ or } \lambda^2 + \mu^2 + \frac{2(d-2)}{d}(1-\lambda)(1-\mu) \leq 1.$$

We consider first the symmetric case $\lambda = \mu$. In this situation, the POVMs $\mathcal{N}_\lambda[A]$ and $\mathcal{N}_\lambda[B]$ are compatible if and only if

$$\lambda \leq \frac{1}{2}\left(1 + \frac{1}{1+\sqrt{d}}\right). \tag{8.9}$$

We shall show that for the same symmetric amount of noise and with a particular choice of an isometry $V : \mathbb{C}^r \to \mathbb{C}^d$, reducing the dimension of two incompatible noisy MUB measurements renders them compatible.

Theorem 8.7.1. *Consider two POVMs A, B corresponding to a pair of mutually unbiased bases which can be extended to a triple of MUBs. For any $2 \leq r < \sqrt{d}$, there exists a non-empty interval $\Lambda_{r,d} \subset [0, 1]$ (see Eq. (8.10)) such that, for all $\lambda \in \Lambda_{r,d}$,*

- *the noisy MUB measurements $\mathcal{N}_\lambda[A]$, $\mathcal{N}_\lambda[B]$ are incompatible*
- *their reduced versions $V^*\mathcal{N}_\lambda[A]V$, $V^*\mathcal{N}_\lambda[B]V$ are compatible,*

where $V : \mathbb{C}^r \to \mathbb{C}^d$ is an isometry obtained by truncating a third MUB.

Before giving the proof of the theorem, note that a triple of MUBs exists in every dimension, see [115, 116].

Proof. Consider a third basis $\{|c_k\rangle\}_{k=1}^d$ of $\mathbb{C}^d$ such that $\{a_i\}$, $\{b_j\}$, and $\{c_k\}$ form a set of three mutually unbiased bases. We define $V : \mathbb{C}^r \to \mathbb{C}^d$ as $V = \sum_{k=1}^r |c_k\rangle\langle k|$; it is clear that V is an isometry.

Note first that the range of parameters λ for which the noisy POVMs $\mathcal{N}_\lambda[A]$, $\mathcal{N}_\lambda[B]$ are incompatible was computed in Eq. (8.9):

$$\frac{1}{2}\left(1 + \frac{1}{1+\sqrt{d}}\right) < \lambda \leq 1.$$

We shall now compute the range of the parameter λ for which we can use Proposition 8.3.5 in its symmetric version for the reduced POVMs $V^*\mathcal{N}_\lambda[A]V$ and $V^*\mathcal{N}_\lambda[B]V$ to certify their compatibility. Let us first calculate, for $i \in [d]$, $\lambda_{\min}(V^*\mathcal{N}_\lambda[A]_iV)$:

$$\lambda_{\min}(V^*\mathcal{N}_\lambda[A]_iV) = \frac{1-\lambda}{d} + \lambda \cdot \lambda_{\min}\left[\sum_{k,l=1}^r \langle c_k|a_i\rangle\langle a_i|c_l\rangle\,|k\rangle\langle l|\right].$$

Note that the operator in the bracket above has unit rank, hence the second term is null. We have thus $\lambda_{\min}(V^*\mathcal{N}_\lambda[A]_iV) = \frac{1-\lambda}{d}$, for all $i \in [d]$. A simple calculation gives

$$\operatorname{Tr} V^*\mathcal{N}_\lambda[A]_iV = \frac{r}{d}.$$

The same calculation can be performed, and the same result is obtained, for $V^*\mathcal{N}_\lambda[B]V$. Putting these together, we find that:

$$\lambda \leq \frac{2+r}{2(1+r)} \implies \begin{cases} \lambda_{\min}(V^*\mathcal{N}_\lambda[A]_iV) \geq \frac{1}{2(1+r)} \operatorname{Tr} V^*\mathcal{N}_\lambda[A]_iV & \forall i \in [d] \\ \lambda_{\min}(V^*\mathcal{N}_\lambda[B]_jV) \geq \frac{1}{2(1+r)} \operatorname{Tr} V^*\mathcal{N}_\lambda[B]_jV & \forall j \in [d], \end{cases}$$

showing that the assumptions of Proposition 8.3.5 hold, and thus that the POVMs $V^*\mathcal{N}_\lambda[A]V$ and $V^*\mathcal{N}_\lambda[B]V$ are compatible for the respective range of λ.

Define now the interval

$$\Lambda_{r,d} := \left(\frac{2+\sqrt{d}}{2(1+\sqrt{d})}, \frac{2+r}{2(1+r)}\right]. \tag{8.10}$$

From the computations above, we know that for all $\lambda \in \Lambda_{r,d}$, the POVMs satisfy the two points in the statement; the interval $\Lambda_{r,d}$ is non-empty as soon as $2 \leq r < \sqrt{d}$. □

Let us now consider the asymmetric version of Theorem 8.7.1, where the amount on white noise added to each POVM can be different. We first introduce a generalization of the compatibility regions from [80, Section III] and [81, Definition 3.32].

Definition 8.7.2. *Given a g-tuple $\mathbf{A}$ of d-dimensional POVMs, we define its* restricted compatibility region *to be the subset*

$$[0,1]^g \ni \Delta(\mathbf{A}; r) = \{\mathbf{s} \in [0,1]^g : \exists V : \mathbb{C}^r \to \mathbb{C}^d \text{ s.t. the reduced POVMs } V^*\mathcal{N}_{s_1}[A^{(1)}]V, V^*\mathcal{N}_{s_2}[A^{(2)}]V, \ldots, V^*\mathcal{N}_{s_g}[A^{(g)}]V \text{ are compatible}\}.$$

Using the generalization of the cloning criterion to asymmetric noise parameters from Theorem 8.3.6, we prove the following lower bound for the compatibility regions $\Delta(\mathbf{A}, r)$ for tuples of MUBs.

Proposition 8.7.3. *For any g-tuple of MUBs $\mathbf{A}$ which can be extended to a $(g+1)$-tuple of MUBs, we have $\Gamma^{clone}(g, r) \subseteq \Delta(\mathbf{A}; r)$.*

Proof. Let $\mathbf{s} \in \Gamma^{clone}(g, r)$, and consider the isometry $V := \sum_{k=1}^{r} |c_k\rangle\langle k|$, where $\{|c_k\rangle\}_{k=1}^{d}$ is the $(g+1)$-th MUB from the statement. To conclude, it is enough to verify the assumptions of Theorem 8.3.6. The computations here are similar to the ones from Theorem 8.7.1. We have, for all $x \in [g]$ and $i \in [d]$,

$$\lambda_{\min}(V^* \mathcal{N}_{s_x}[A^{(x)}]_i V) = \frac{1 - s_x}{d}$$
$$\mathrm{Tr}(V^* \mathcal{N}_{s_x}[A^{(x)}]_i V) = \frac{r}{d}.$$

Hence,

$$s_x = 1 - \min_{i \in [k_x]} \frac{d\lambda_{\min}(V^* \mathcal{N}_{s_x}[A^{(x)}]_i V)}{\mathrm{Tr}(V^* \mathcal{N}_{s_x}[A^{(x)}]_i V)}$$

satisfies the hypothesis of Theorem 8.3.6. □

We leave the question of deriving upper bounds for the sets $\Delta(\mathbf{A}, r)$ open.

8.8 Algebraic considerations

A simple way of using dimension reduction to render incompatible measurements compatible is to ensure that, after the reduction, the POVM elements of the measurements are commutative. Moreover, in the case of 2 POVMs, one can push this idea even further and render one of the reduced POVMs trivial, ensuring thus compatibility. The overarching theme of this section is to use the two algebraic characterizations of compatibility (commutativity and trivial POVMs) to obtain very general dimension reduction results. The price to pay for this generality is that, for some very specific situations, the results can be relatively weak, when compared with more specialized techniques, such as the ones from Sections 8.6 and 8.7.

We start with a dimension reduction method by which POVMs are rendered commutative (and thus compatible). The following construction has been introduced in [117, Theorem 3] and further refined in [118, Proposition 2.4]. The connection with quantum error correction can be understood as follows: on the code space, the POVM channels act like the identity (up to a scalar), hence the reduced POVMs are trivial.

For the sake of completeness, we recall it here in full details and adapt it to our setting, emphasizing the intermediate step related to commutative POVMs.

Definition 8.8.1. *For a g-tuple of POVMs $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$ on $\mathcal{M}_d$, we define their* commutativity dimension *as*

$$T(\mathbf{A}) := \max\{r \in [d] : \exists V : \mathbb{C}^r \to \mathbb{C}^d \text{ isometry s.t.}$$
$$\forall x \neq y \in [g], \forall i \in [k_x], \forall j \in [k_y], \quad [V^* A_i^{(x)} V, V^* A_j^{(y)} V] = 0\}.$$

We recall the following result from [118], showing that tuples of matrices can be reduced to commutative operators, when the dimension is large enough.

Proposition 8.8.2. *Consider m self-adjoint $d \times d$ matrices $A_1, \ldots, A_m$ and let*

$$n + 1 = \dim \operatorname{span}_{\mathbb{R}}\{A_1, \ldots, A_m, I_d\}.$$

If $d \geq (n+1)(r-1)$, then there exist r orthonormal vectors $x_1, \ldots, x_r \in \mathbb{C}^d$ such that, for all $s \in [m]$, $\langle x_i | A_s x_j \rangle = 0$, whenever $i \neq j \in [r]$. In other words, the matrices A are diagonal when restricted to the span of the vectors $\{x_1, x_2, \ldots, x_r\}$.

Proof. One can observe that if $\{B_1, \ldots, B_n\}$ is a basis of $\operatorname{span}_{\mathbb{R}}\{A_1, \ldots, A_m, I_d\}$, then there exist r orthogonal vectors $x_1, \ldots, x_r$ such that $\langle x_i | A_s x_j \rangle = \lambda_s \delta_{ij}$ for all $i, j \in [r]$ and $s \in [m]$ iff the same holds true for the matrices $B_1, \ldots, B_n$. The result follows then from the first part of [118, Proposition 2.4]. □

We shall now use the result above for the set of effects of a g-tuple of POVMs, to find an isometry reducing them to commuting POVMs. The following theorem combines Definition 8.8.1 with the lower bound from Proposition 8.8.2.

Theorem 8.8.3. *Consider a g-tuple $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$, where $A^{(x)} = (A^{(x)}_1, \ldots, A^{(x)}_{k_x})$ is a POVM with k_x outcomes. Let*

$$n + 1 := \dim \operatorname{span}_{\mathbb{R}}\{A^{(x)}_i\}_{x \in [g], i \in [k_x]} \leq 1 - g + \sum_{x=1}^{g} k_x.$$

Then, we have the following lower bound:

$$R(\mathbf{A}) \geq T(\mathbf{A}) \geq 1 + \left\lfloor \frac{d}{n+1} \right\rfloor \geq 1 + \left\lfloor \frac{d}{1 - g + \sum_{x=1}^{g} k_x} \right\rfloor. \tag{8.11}$$

Proof. For any $r \leq T(\mathbf{A})$, there exists an isometry $V : \mathbb{C}^r \to \mathbb{C}^d$ such that the reduced effect operators $V^* A^{(x)}_i V \in \mathcal{M}_r$ commute with $V^* A^{(y)}_j V$ for all i, j and $x \neq y$. In particular, the reduced POVMs $V^* A^{(x)} V$ are compatible: $R(\mathbf{A}) \geq T(\mathbf{A})$. The second and third inequalities in (8.11) follow from Proposition 8.8.2. □

Remark 8.8.4. *In the case where $n \geq d$, the lower bound (8.11) is trivial.*

Remark 8.8.5. *In the definition of $T(\mathbf{A})$ we only ask that reduced effects from different POVMs commute, while the use of Proposition 8.8.2 guarantees that all the reduced effects commute. It would be interesting to find out whether one can gain something by exploiting this fact.*

Let us illustrate the previous result by the following striking corollary, corresponding to the case $d = 3$, $g = 2$, $k_1 = k_2 = 2$.

Corollary 8.8.6. *Any pair of qutrit effects can be reduced to a pair of commuting (and thus compatible) qubit effects.*

Example 8.8.7. *Let us consider the following two qutrit effects, built from the computational and the Fourier bases in $\mathbb{C}^3$:*

$$E = |1\rangle\langle 1| + \frac{|2\rangle\langle 2|}{2} \qquad F = |f_1\rangle\langle f_1| + \frac{|f_2\rangle\langle f_2|}{2},$$

where $f_{1,2,3}$ are the columns of the Fourier matrix

$$F_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix},$$

with $\omega = \exp(2\pi \mathrm{i}/3)$*, see also Example 8.6.2. The fact that the effects* E, F *are incompatible (that is, the POVMs* $(E, I_3 - E)$ *and* $(F, I_3 - F)$ *are incompatible) follows from the following semidefinite program [70]:*

$$\begin{aligned} \textit{minimize} \quad & \lambda \\ \textit{subject to} \quad & X \geq 0 \\ & X \leq E \\ & X \leq F \\ & \lambda I_3 + X \geq E + F. \end{aligned}$$

In the SDP above, the variable X *corresponds to the single free value of a joint POVM for* E, F*. The effects* E, F *are compatible if and only if the value of the SDP above is smaller or equal than one [9, Eq. (4)]. For our choice of* E, F*, it can be seen numerically that the value of the program is* ≈ 1.577*, certifying the incompatibility of* E *and* F*.*

We choose the isometry

$$V = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{\omega}{\sqrt{2}} \end{bmatrix},$$

for which the reduced effects read

$$V^* E V = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \qquad \textit{and} \qquad V^* F V = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

The reduced effects are commutative, hence compatible.

We now move on to another method by which incompatible POVMs can be rendered compatible by dimension reduction. This time, we shall consider a single POVM and "trivialize" it by reducing it with an isometry. In the language of error correction, we are constructing a subspace of the Hilbert space on which the measurement channel acts like the identity.

Definition 8.8.8. *Given a single POVM* A *with* k *outcomes on* $\mathcal{M}_d$*, its* scalar dimension *is*

$$S(A) := \max\{r \in [d] \, : \, \exists V : \mathbb{C}^r \to \mathbb{C}^d \textit{ isom. s.t. } \forall i \in [k], \quad V^* A_i V \sim I_r\}.$$

The definition above is related to the notion of *higher rank (joint) numerical range* introduced in [119] for one matrix and generalized in [118] for several matrices. We recall the following lower bound from [118, Proposition 2.4], which uses Tverberg's theorem [120] (see also [121]) to render the diagonal matrices from Proposition 8.8.2 multiples of the identity.

Proposition 8.8.9. *Consider* m *self-adjoint* $d \times d$ *matrices* $A_1, \ldots, A_m$ *and let*

$$n + 1 = \dim \operatorname{span}_{\mathbb{R}}\{A_1, \ldots, A_m, I_d\}.$$

If $d \geq (n+1)^2(r-1)$*, then there exist* r *orthonormal vectors* $x_1, \ldots, x_r \in \mathbb{C}_d$ *such that, for all* $s \in [m]$*, there exists a scalar* $\lambda_s \in \mathbb{R}$ *such that* $\langle x_i | A_s x_j \rangle = \delta_{ij} \lambda_s$*, for all* $i, j \in [r]$*.*

Proof. The statement follows from the second part of the proof of [118, Proposition 2.4], by making the same observation as in the proof of Proposition 8.8.2. □

We can gather the results above in the following theorem.

Theorem 8.8.10. *Consider a pair of POVMs A, B on $\mathcal{M}_d$. Let k be the number of outcomes of the POVM A, and define*

$$n + 1 := \dim \operatorname{span}_{\mathbb{R}}\{A_i\}_{i\in[k]} \leq k.$$

We have the following lower bound:

$$R(A,B) \geq S(A) \geq 1 + \left\lfloor \frac{d}{(n+1)^2} \right\rfloor \geq 1 + \left\lfloor \frac{d}{k^2} \right\rfloor. \tag{8.12}$$

Proof. For any $r \leq S(A)$ (resp. $r \leq S(B)$), there exists an isometry $V : \mathbb{C}^r \to \mathbb{C}^d$ such that the POVM V^*AV (resp. V^*BV) is trivial. In particular, the POVMs V^*AV and V^*BV are compatible, and thus $R(A,B) \geq \max(S(A), S(B))$. The second inequality in (8.12) follows from Proposition 8.8.9. □

Remark 8.8.11. *In the case where $(n+1)^2 \geq d$, the lower bound (8.12) is trivial. In particular, if a POVM A has k linearly independent effects and $k > \sqrt{d}$, the bound (8.12) is trivial. Hence, Theorem 8.8.10 is useful for POVMs with few outcomes.*

Example 8.8.12. *Going back to the two qubit effects from Example 8.8.7, note that the reduced POVM $(V^*FV, I_2 - V^*FV)$ is the trivial POVM $(I_2/2, I_2/2)$.*

To conclude, using ideas from the theory of quantum error correction, we have given in this section two lower bounds on the compatibility dimension of a tuple of POVMs $\mathbf{A}$:

- a first one in terms of the *commutativity dimension* $T(\mathbf{A})$ of the tuple, Theorem 8.8.3;
- a second one in terms of the *scalar dimensions* $S(A)$ and $S(B)$ of any pair POVMs (A, B), see Theorem 8.8.10.

We would like to point out that these very general results are useful in the regime where the POVMs have few outcomes (or, rather, the span of the effect operators is low-dimensional). The results in this section cannot be applied, for example, to the cases of (noisy) orthonormal bases that were studied in Sections 8.6, 8.7.

8.9 Dimension dependent bounds and spin systems

We prove in this section results for isometry-independent reductions, corresponding to the notion of strong compatibility dimension from Definition 8.4.4.

We recall the following compatibility criterion from [80, Section VIII] and [81, Section 7] which guarantees the compatibility of noisy versions of POVMs, with a noise parameter depending on the dimension of the Hilbert space, and independent of the number of measurements. We shall explicitly consider separately the case of 2-outcome (or dichotomic) POVMs, with the example of maximally incompatible *spin system measurements* in mind.

Proposition 8.9.1. *[80, Corollary VIII.4] and [81, Theorem 7.1] Let $A^{(1)}, \ldots, A^{(g)}$ be g arbitrary 2-outcome POVMs on $\mathcal{M}_d$. Then, their noisy versions $\tilde{A}^{(x)}$ are compatible, where*

$$\tilde{A}_i^{(x)} = \mathcal{N}_{1/(2d)}[A^{(x)}]_i = \frac{1}{2d} A_i^{(x)} + \left(1 - \frac{1}{2d}\right) \frac{I_d}{2}. \tag{8.13}$$

More generally, consider a g-tuple $(B^{(x)})_{x=1}^g$, where $B^{(x)}$ is a k_x-valued POVM on $\mathbb{C}^d$. Then, their noisy versions $\tilde{B}^{(x)}$ are compatible, where

$$\tilde{B}_i^{(x)} = \mathcal{N}_{1/(2d(k_x-1))}[B^{(x)}]_i = \frac{1}{2d(k_x-1)} B_i^{(x)} + \left(1 - \frac{1}{2d(k_x-1)}\right) \frac{I_d}{k_x}. \tag{8.14}$$

This compatibility criterion is of particular interest in the setting of our work, given the dimension dependence of the noise parameters in the equations (8.13) and (8.14). Note that for small values of g, the compatibility result above can be seen to follow from other type of arguments, such as cloning [98]. We obtain the following universal lower bound on the quantity $\bar{R}(\cdot)$ from Definition 8.4.4, giving thus the first lower bound on the strong compatibility dimension.

Theorem 8.9.2. *Let $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$ be a g-tuple of 2-outcome POVMs on $\mathcal{M}_d$. Then, for all $1 \leq r \leq d$ and $t \in [0, 1/(2r)]$, we have $\bar{R}(\mathcal{N}_t[\mathbf{A}]) \geq r$.*

More generally, consider a g-tuple $\mathbf{B} = (B^{(1)}, \ldots, B^{(g)})$, where $B^{(x)}$ is a k_x-valued POVM on $\mathbb{C}^d$. Then, for all $1 \leq r \leq d$ and $\mathbf{t} \in [0,1]^g$ such that $t_x \leq 1/(2r(k_x - 1))$, we have $\bar{R}(\mathcal{N}_{\mathbf{t}}[\mathbf{B}]) \geq r$.

Proof. Let us prove the more general statement about the g-tuple $\mathbf{B}$. Fix an integer r and a vector $\mathbf{t}$ as in the statement. Consider also an arbitrary isometry $V : \mathbb{C}^r \to \mathbb{C}^d$. From Lemma 8.2.4, we have that, for all $x \in [g]$,

$$V^* \mathcal{N}_{t_x}[B^{(x)}] V = \mathcal{N}_{t_x}[V^* B^{(x)} V].$$

Using Proposition 8.9.1 and the condition on the vector $\mathbf{t}$, we infer that the POVMs $\mathcal{N}_{\mathbf{t}}[V^* \mathbf{B} V]$ are compatible, proving the claim. □

Let us now use the previous result to obtain bounds on the strong compatibility dimension of spin system measurements, which we introduce next. From a physical point of view [122, Section 5.4], it was discovered by Dirac that the spin property appears naturally in his equation when he was searching for a relativistic quantum equation of electrons. In his equation the Clifford algebra appears as a particular representation of the homogeneous Lorentz group. Since this representation contains naturally the spin one-half representation described by the Pauli matrices, his equation presents the conceptual and the natural description of the spin as a fundamental property. Mathematically, spin systems are sets of anti-commuting, self-adjoint, unitary operators. The paradigmatic example of such operators are the Pauli matrices $\sigma_{X,Y,Z} \in \mathcal{M}_2(\mathbb{C})$. Higher level spin systems are defined recursively, as follows. At level $k = 0$, we have a single matrix,

$$F_1^{(0)} := [1] \in \mathcal{M}_1(\mathbb{C}).$$

At level $k = 1$, we have the Pauli matrices:

$$F_1^{(1)} = \sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad F_2^{(1)} = \sigma_Y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \quad \text{and} \quad F_3^{(1)} = \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For larger levels, define recursively the matrices of size 2^{k+1}

$$F_i^{(k+1)} = \sigma_X \otimes F_i^{(k)} \; \forall i \in [2k+1] \quad \text{and} \quad F_{2k+2}^{(k+1)} = \sigma_Y \otimes I_{2^k}, \quad F_{2k+3}^{(k+1)} = \sigma_Z \otimes I_{2^k}.$$

For example, at level 2, we have the five matrices

$$F_1^{(2)} = \sigma_X \otimes \sigma_X, \qquad F_2^{(2)} = \sigma_X \otimes \sigma_Y, \qquad F_3^{(2)} = \sigma_X \otimes \sigma_Z, \qquad F_4^{(2)} = \sigma_Y \otimes I_2, \qquad F_5^{(2)} = \sigma_Z \otimes I_2.$$

From the $2k+1$ matrices at level k, we construct $2k+1$ dichotomic POVMs

$$A_1^{(x)} = (I_{2^{k+1}} + F_x)/2 \qquad A_2^{(x)} = (I_{2^{k+1}} - F_x)/2, \qquad x \in [2k+1].$$

We recall the following result from [80] regarding the noise robustness of the tuple $\mathbf{A} = (A^{(x)})_{x \in [2k+1]}$; note that the same result was derived in the symmetric case in [123].

Proposition 8.9.3. *[80, Section VIII.B] For every $k \geq 1$, the $(2k+1)$-tuple of 2-outcome POVMs $\mathcal{N}_t[\mathbf{A}]$ acting on $\mathbb{C}^{2^{k+1}}$ is compatible if and only if $\|t\|_2 \leq 1$.*

Combining the previous result with Theorem 8.9.2, we obtain the following result, stating that, for appropriate noise parameters, the strong compatibility dimension of a noisy spin system POVM is neither 1 nor maximal. In other words, the noisy spin system POVMs are not compatible, but all reductions to a non-trivial fixed dimension become compatible.

Proposition 8.9.4. *For any $r \geq 2$, $k \geq 2r^2 + 1 \geq 9$, and all $t \in (1/\sqrt{2k+1}, 1/(2r)]$, the spin system POVMs* $\mathbf{A}$ *at level k satisfy*

$$r \leq \bar{R}(\mathcal{N}_t[\mathbf{A}]) \leq 2^{k+1} - 1.$$

Proof. The statement about compatibility follows from $t \leq 1/(2r)$ and Theorem 8.9.2. The incompatibility statement follows from Proposition 8.9.3 and

$$t > \frac{1}{\sqrt{2k+1}} \implies \|t(\underbrace{1, 1, \ldots, 1}_{2k+1 \text{ times}})\|_2 > 1.$$

The inequality between k and r ensures the existence of noise parameters for which the interval in the statement is non-empty. □

8.10 Conclusion

In this paper, we have introduced a new measure of the incompatibility of a pair (or a tuple) of quantum measurements. The *compatibility dimension* of a set of POVMs is the maximal dimension of a Hilbert space to which the restrictions of the given measurements are compatible. A related notion, that of the *strong* compatibility dimension is defined in a similar manner, but requiring that the restrictions to *all* Hilbert subspaces of that given dimension are compatible.

We then proceed to analyze the properties of these quantities, relating them to (in-)compatibility criteria. We study several examples in details, such as pairs of von Neumann measurements and mutually unbiased bases. We also provide lower bounds for these quantities using constructions inspired from the theory of error correcting codes.

Several questions are left open. Importantly, good upper bounds on the (strong) compatibility dimensions are lacking. One would equally like to compute exactly these dimensions in very simple cases, such as the measurements in the computational basis and the one in the Fourier basis. The optimality of the algebraic techniques used in Sections 8.6 (the quantity $\mathcal{Z}(U)$) and 8.8 is also left open.

Chapter 9

Measurement incompatibility vs. Bell nonlocality:an approach via tensor norms

This Chapter is a reproduction of the paper [2].

Measurement incompatibility and quantum nonlocality are two key features of quantum theory. Violations of Bell inequalities require quantum entanglement and incompatibility of the measurements used by the two parties involved in the protocol. We analyze the converse question: for which Bell inequalities is the incompatibility of measurements enough to ensure a quantum violation? We relate the two questions by comparing two tensor norms on the space of dichotomic quantum measurements: one characterizing measurement compatibility and the second one characterizing violations of a given Bell inequality. We provide sufficient conditions for the equivalence of the two notions in terms of the matrix describing the correlation Bell inequality. We show that the CHSH inequality and its variants are the only ones satisfying it.

9.1 Introduction

Since its discovery, quantum mechanics was formalized as a theory with many foundational aspects which differ significantly from classical mechanics. Some of these deep questions, and their relation among them are still subject to investigation nowadays. Understanding these notions and their interplay is crucial for the development of the second quantum revolution.

Two of the most important conceptual revolutions put forward by quantum mechanics are the notions of *nonlocality of correlations* and the *incompatibility of quantum measurements*. The latter notion, that of measurement incompatibility is one of the most unintuitive aspects of the quantum world, when examined from a classical perspective: there exist (quantum) measurements which cannot be performed simultaneously on a given quantum system.

It is well-known that quantum nonlocality is one of the fundamental aspects of quantum theory that gives rise to a lot of questions about quantum reality. John Bell [124] gave a complete answer to the debate about the nonlocality and elucidated the intrinsic probabilistic aspect of quantum theory. The answer he provides is that any local theory must obey some inequality, while if one applies the predictions of quantum mechanics, the aforementioned inequality can be violated. This means that the quantum world is completely non-local, which, in turn, means that there are phenomena that we could not understand with our classical macroscopic point of view. Such conclusion provides a complete answer about the intrinsic reality of the quantum world. Such violations of correlation inequalities were completely confirmed experimentally in Alain Aspect's experiment [125], and in a loophole-free manner in [43].

In the modern language of quantum information theory, such correlation inequalities can be understood as *non-local games* [126]. In such a game, two players, called traditionally Alice and Bob, play cooperatively against a Referee. Alice and Bob are space-like separated, hence once the game starts they can no longer communicate. However, they both know the rules of the game, and they can meet before the game starts and make a strategy. Technically, the games we consider are defined by a matrix M, which encodes the pay-off the players receive; in particular, we shall consider in this work exclusively XOR games with N questions and two answers.

Two scenarios are of particular importance for us: Alice and Bob are either allowed to use *classical strategies* (where they share classical randomness) or *quantum strategies* (where they share a bipartite entangled quantum state). It turns out that the optimal probabilities to win the game with classical or, respectively, quantum strategies, can be formulated as two different *tensor norms* of the matrix that encodes the rules of the game (seen as a $2-$tensor). This is one of the instances where tensor norms (and Banach space theory in general) has found applications in the theory of non-local games.

The main goal of the current work is to relate, in a *quantitative manner* the notion of *measurement incompatibility* to that of *Bell inequality violations* in a very general setting. Our original motivation was the seminal work [9], where the authors connected, in a qualitative manner, the incompatibility of Alice's measurements in the CHSH game, with possible violations of the CHSH inequality. Our results can be seen to build on this example, generalizing it in two different directions:

- we go beyond the CHSH game, allowing all (reasonable) correlation XOR games
- we relate, in a quantitative manner, the largest possible violation of a Bell inequality to the incompatibility robustness of Alice's measurements.

In order to achieve these goals, our framework is different than the usual setting of non-local games, in the respect that

Alice's dichotomic measurements are fixed.

Optimizing over Bob's choice of N dichotomic measurements and over the players' shared entangled state, we can express the quantum bias of the given non-local game M as a tensor norm of Alice's N-tuple of measurements, which we denote by $\|A\|_M$. In particular, with Alice's choice of measurements fixed to be A, the players will violate the Bell inequality corresponding to M if and only if $\|A\|_M > \beta(M)$, where $\beta(M)$ is the classical bias of the game.

On the quantum measurement (in-)compatibility side, we revisit the construction from [86], where the compatibility of a N-tuple of dichotomic quantum measurements has been described with the help of a tensor norm, dubbed $\|A\|_c$. We give a direct proveof the result showing that N dichotomic measurements $A = (A_1, \ldots, A_N)$ are compatible iff $\|A\|_c \leq 1$. The value of the compatibility norm $\|\cdot\|_c$ is related to the notion of *compatibility robustness*: the value of the norm is precisely the quantity of (white) noise one needs to add to the tuple of dichotomic measurements in order to render them compatible.

Having formulated the key physical principles of this work (quantum incompatibility and Bell nonlocality), we get now to our main point: the relation between them. This question has received already a lot of attention in the literature. The starting point is the equivalence first observed in [9]: for the CHSH game [127] with two questions, Alice's pair of measurements are incompatible if and only if there exists an entangled state and a choice for Bob's pair of measurements such that they can obtain a violation of the CHSH Bell inequality. It is equally well-known that the two notions are not equivalent in more general situations, see [89].

In this work, we provide a definitive answer to this question, using the framework of tensor norms. More precisely, we express the following quantities as tensor norms:

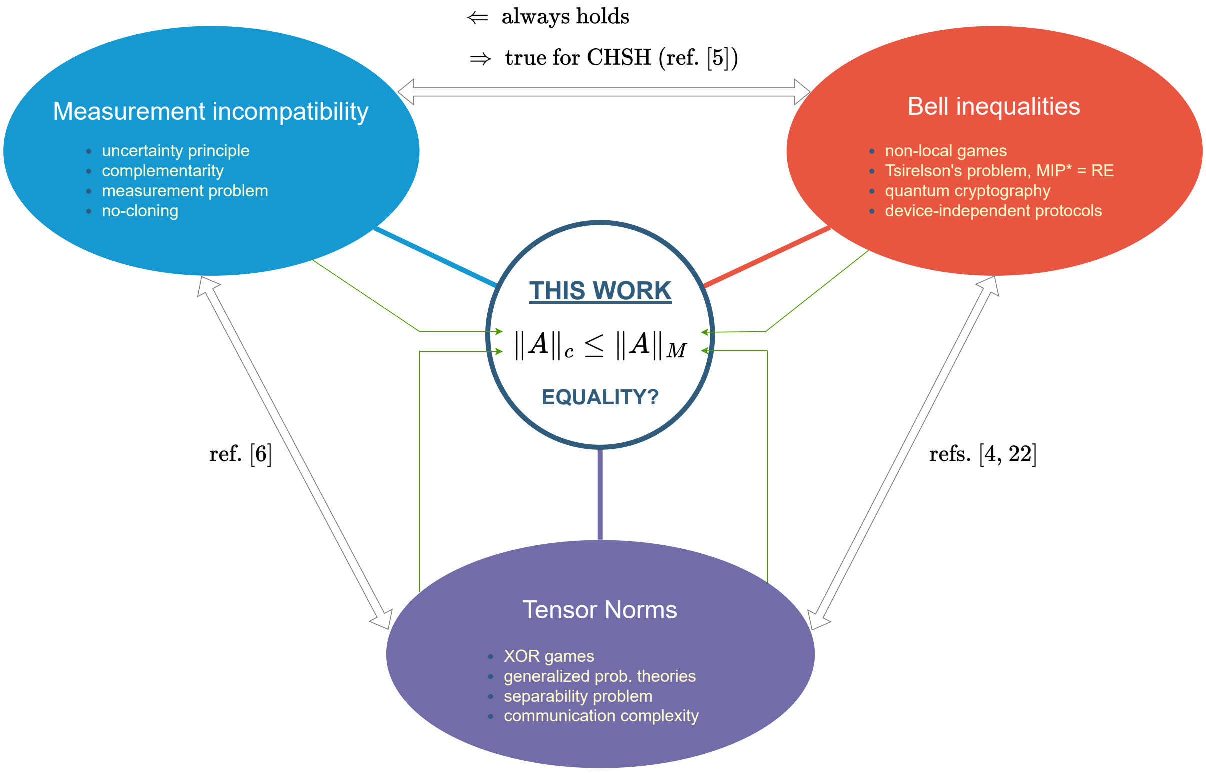


Figure 9.1: In this work, we relate measurement incompatibility with Bell inequalities using the formalism of tensor norms. Pairwise connections having already been established in the literature, we bring the three concepts together for the first time.

- *incompatibility*: how much (white) noise one needs to add to a tuple of dichotomic POVMs to render them compatible
- *quantum bias of a correlation game*: what is the maximal value of the game (normalized to have classical bias 1), when Alice's tuple of dichotomic measurements are fixed.

We then discuss how these norms compare, and when they are equal see Figure 9.1. We provide sufficient conditions for equality, and then show that only the CHSH game (and its permutations) satisfy them, emphasizing the special role of the CHSH inequality.

Our paper is organised as follows. In Section 9.2 we (informally) state the main results of our paper and their interpretation. In Section 9.3 we recall the notion of compatibility for quantum measurements. We present in Section 9.4 the basic definitions of tensor norms from Banach space theory, focusing on the examples needed in this work. In Section 9.5 we introduce the framework of Bell nonlocality as non-local games and relate the values of these games to tensor norms. In Section 9.6 we introduce the main definition of the nonlocality norm $\|A\|_M$ that will characterise the violation of the Bell inequality. In Section 9.7 we introduce the compatibility norm $\|A\|_c$ that will characterise the compatibility of Alice's measurement. We present in Section 9.8 our main theorems, discussing also under which conditions the violation of a Bell inequality implies measurement incompatibility. In our framework, we provide a conceptual explanation of the main result in [9], and we also analyze new Bell inequalities, such as different deformations of the CHSH inequality and the pure correlation part of the I_{3322} tight Bell inequality; for the latter, the two notions are not equivalent, as noticed in [89].

9.2 Main results

In this section, we introduce the main definitions and the main results of our work. Our goal is to unify two fundamental notions of quantum theory, *measurement incompatibility* and *Bell inequality violations*. To do so, we shall work in the framework of *non-local games*, where the rules of a correlation game are encoded in a real $N \times N$ matrix M, and *Alice's dichotomic measurements are fixed.* Note that in this work we shall be considering general (not necessarily projective) measurements, mathematically encoded by POVMs.

The maximum value of the game M, when Alice's measurements are fixed, is given by the following quantity.

Definition 9.2.1 (The M-Bell-locality tensor norm)**.** *Let M an invertible* Bell functional *and Alice's N-tuple of dichotomic measurements $A = (A_1, \ldots, A_N)$, we define the following tensor norm:*

$$\|A\|_M := \sup_{\|\psi\|=1} \sup_{\|B_y\| \leq 1} \left\langle \psi \middle| \sum_{x,y=1}^{N} M_{xy}\, A_x \otimes B_y \middle| \psi \right\rangle = \lambda_{\max} \left[\sum_{y=1}^{N} \left| \sum_{x=1}^{N} M_{xy}\, A_x \right| \right].$$

The quantity $\|A\|_M$ is the maximum value of the game M, when optimizing over quantum strategies, with Alice's measurements being fixed. The measurements $A = (A_1, \ldots, A_N)$ are called *M-Bell-local* there is no violation of the Bell inequality corresponding to M: $\|A\|_M \leq \beta(M)$, with $\beta(M)$ being *the classical bias* of the game (which, importantly, can also be expressed as a tensor norm). If this is not the case, we call Alice's measurements *M-Bell-non-local.*

Regarding compatibility, we are concerned with the same question as before: are Alice's dichotomic measurements compatible or not? The following quantity was introduced, in the more abstract setting of generalized probabilistic theories in [86], see also [87].

Definition 9.2.2 (The compatibility tensor norm)**.** *For a tensor $A \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$, we define the following quantity:*

$$\|A\|_c := \inf \left\{ \left\| \sum_{j=1}^{K} H_j \right\|_\infty \, : \, A = \sum_{j=1}^{K} z_j \otimes H_j, \;\; s.t. \;\; \forall j \in [K], \, \|z_j\|_\infty \leq 1 \text{ and } H_j \geq 0 \right\}.$$

The compatibility norm, together with the injective tensor product of ℓ_∞ and S_∞ norms, completely characterize compatibility of tuples of dichotomic quantum measurements [86, 87].

Proposition 9.2.3. *Let $A = (A_1, \ldots, A_N)$ be a N-tuple of self-adjoint $d \times d$ complex matrices. Then:*

1. *A is a collection of dichotomic quantum observables (i.e. $\|A_i\|_\infty \leq 1$ $\forall i$) if and only if $\|A\|_\varepsilon \leq 1$.*
2. *A is a collection of* compatible *dichotomic quantum observables if and only if $\|A\|_c \leq 1$.*

The compatibility norm allows Alice to know whether her measurements are compatible ($\|A\|_c \leq 1$) or not ($\|A\|_c > 1$); in the latter case, the the minimal quantity of white noise that needs to be mixed in the measurements in order to render them compatible is $1/\|A\|_c$, providing an operational interpretation of the compatibility norm.

To sum up, in the setting of tensor norms,

- Alice's measurements are *M-Bell-local* if and only if $\|A\|_M \leq \beta(M) = \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N}$.
- Alice's measurements are compatible if and only if $\|A\|_c \leq 1$.

To understand the relation between nonlocality and compatibility, we now have to compare the two norms $\|\cdot\|_c$ and $\|\cdot\|_M$.

Theorem 9.2.4. *Consider an N-input 2-output non-local game M, corresponding to a matrix $M \in \mathcal{M}_N(\mathbb{R})$. Then, for any N-tuple of self-adjoint matrices $A = (A_1, \ldots, A_N)$, we have*

$$\|A\|_M \leq \|A\|_c \, \beta(M) = \|A\|_c \, \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N}.$$

In particular, if Alice's measurements A are M-Bell-non-local, then they must be incompatible.

In the theorem above, we have upper bounded the M-Bell-locality norm by the compatibility norm that depends only on Alice's measurement times the classical bias of the game. This inequality is a *quantitative* version of the well-known, *qualitative* fact that if Alice's measurements are compatible, she will never observe any Bell inequality violation (i.e. her measurements are M-Bell-local).

One of our main contributions is to raise and answer the converse question: we want to upper bound the compatibility norm by the M-Bell-locality norm. In physical terms, we are asking whether, given a Bell inequality M and a tuple of measurements, can Alice observe violations of M using her measurements? We have the following theorem, providing a (partial) answer to this question.

Theorem 9.2.5. *Let $M \in \mathcal{M}_N(\mathbb{R})$ be an invertible matrix. Then, for any N-tuple of self-adjoint matrices $A = (A_1, A_2, \ldots, A_N)$, we have*

$$\|A\|_c \leq \|A\|_M \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}.$$

In the main theorems succinctly stated above, we have compared the compatibility tensor norm and the M-Bell-locality norm. It was shown in [9] that for the CHSH game, the incompatibility of one party's quantum measurements and the violation of a Bell inequality are equivalent. In our setting, this equivalence can be understood as an *equality* of the compatibility norm and the M-Bell-locality norm for M_{CHSH}: we have

$$\| \cdot \|_c = \| \cdot \|_{M_{\text{CHSH}}}.$$

Having restated this classical result in terms of an equality of tensor norms, it is natural to ask whether this equality goes beyond the case of the CHSH inequality. Incompatibility and Bell nonlocality are not, in general, equivalent, as it was shown in [90, 128].

From the main theorems above, any game M must satisfy $\|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \cdot \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \geq 1$. If one wants to conclude $\| \cdot \|_c = \| \cdot \|_M$ from these results, one needs to investigate the equality case in the aforementioned inequality. We show that for any real and invertible matrix M, the following holds.

Proposition 9.2.6. *For any real and invertible matrix M, we have:*

$$\|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \geq \frac{N}{\rho(\ell_\infty^N, \ell_\infty^N)} \geq \sqrt{\frac{N}{2}} \geq 1.$$

For $N \geq 3$, the last inequality above is strict.

The case $N = 2$ needs to be treated separately. We show that for $N = 2$ questions, the only games achieving equality are the CHSH game and variants thereof. We summarize this in the following theorem.

Theorem 9.2.7. *The only invertible non-local game $M \in \mathcal{M}_N(\mathbb{R})$ satisfying*

$$\|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} = 1$$

have two questions ($N = 2$) and are variants of the CHSH game: $M = aM_{CHSH}$ for some $a \neq 0$.

9.3 Compatibility of quantum measurements

This section contains the main definitions and results from the theory of quantum measurements, with the focus on (in-)compatibility and noisy measurements.

In Quantum Mechanics, a system is described by Hilbert space $\mathcal{H}$. Here, we shall consider only finite-dimensional Hilbert spaces: $\mathcal{H} \cong \mathbb{C}^d$, for a positive integer d, which corresponds to the number of degrees of freedom. For example, quantum bits (qubits) are described by the space $\mathbb{C}^2$. Quantum states are formalized mathematically by *density matrices*:

$$\mathcal{M}_d^{1,+} := \{\rho \in \mathcal{M}_d \,:\, \rho \geq 0 \text{ and } \operatorname{Tr} \rho = 1\},$$

where $\mathcal{M}_d$ is the vector space of $d \times d$ complex matrices. Density matrices are positive semidefinite, a relation denoted by $\rho \geq 0$.

Let us now discuss measurements in Quantum Mechanics. Historically, quantum measurements were modelled by *observables*: Hermitian operators acting on the system Hilbert space. The possible outcomes of the measurement are the eigenvalues of the observable, while the probabilities of occurrence are given by the celebrated *Born rule*. This formalism not only allows to obtain the probabilities of the different outcomes (via the Born rule), but also the post-measurement state of the quantum system (the *wave function collapse*). In the current research, we are only concerned with the former, and thus we shall use the more general framework of Positive Operator Valued Measures (POVMs) [129]. We shall write $[n] := \{1, 2, \ldots, n\}$ for the set of the first n positive integers.

Definition 9.3.1. *A* positive operator valued measure *(POVM) on $\mathcal{M}_d$ with k outcomes is a k-tuple $A = (A_1, \ldots, A_k)$ of self-adjoint operators from $\mathcal{M}_d$ which are positive semidefinite and sum up to the identity:*

$$\forall i \in [k], \quad A_i \geq 0 \qquad \text{and} \qquad \sum_{i=1}^{k} A_i = I_d.$$

When measuring a quantum state ρ with the apparatus described by A, we obtain a random outcome from the set $[k]$:

$$\forall i \in [k], \qquad \mathbb{P}(\text{outcome} = i) = \operatorname{Tr}[\rho A_i].$$

The vector of outcome probabilities $(\operatorname{Tr}[\rho A_i])_{i=1}^k$ is indeed a probability vector; note that the properties of the operators A_i, called *quantum effects*, are tailor made for this. This mathematical formalism used to described quantum measurements (or POVMs, or *meters*) does not account for what happens with the quantum particle after the measurement. One can think that the particle is destroyed in the process of measurement (see Figure 9.2) and thus only the outcome probabilities are relevant.

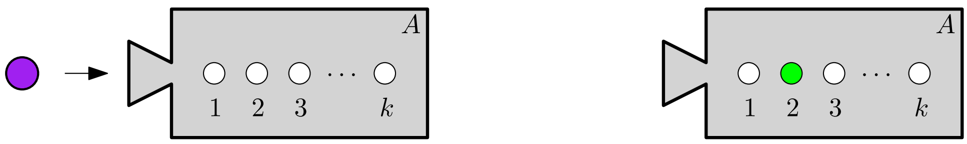


Figure 9.2: Diagrammatic representation of a quantum measurement apparatus. The device has an input canal and a set of k LEDs which will turn on when the corresponding outcome is achieved. After the measurement is performed, the particle is destroyed, and the apparatus displays the classical outcome (here, 2).

Several important classes of POVMs will be discussed in this paper:

- *von Neumann measurements*, where $A_i = |a_i\rangle\langle a_i|$, $i \in [d]$, for an orthonormal basis $\{|a_i\rangle\}_{i=1}^d$ of $\mathbb{C}^d$;
- *trivial measurements*, where the matrices A_i are scalar multiples of the identity: $A_i = p_i I_d$, for some probability vector $p = (p_1, p_2, \ldots, p_k)$.

Let us now define the notion of *compatibility* for quantum measurements, which is central to this paper. Historically, in the physics literature, the notion of compatbility was closely related to that of commutativity of the quantum observables [130, 131]; indeed, sharp POVMs are compatible if and only if the corresponding observables commute. In the modern setting, suppose we want to measure two different physical quantities (modelled by two POVMs A and B) on a given quantum particle in a state ρ. Having at our disposal just one copy of the particle, we cannot, in general, measure simultaneously A and B. However, one can *simulate* measuring A and B on ρ with the help of a third POVM C, by *classically* post-processing the output of C to a pair of outcomes (i, j) for A, respectively B, see Figure 9.3. Importantly, there are many pairs of POVMs A and B for which there is no such C, like the position and momentum operators of a particle in one dimension: it is impossible to attribute simultaneously an exact value to both position and momentum observables.

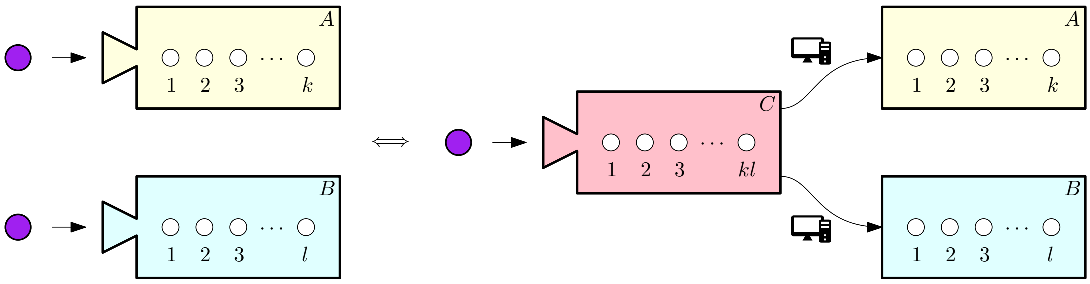


Figure 9.3: The joint measurement of A and B is simulated by by a third measurement C, followed by classical post-processing.

Mathematically, we can either consider general post-processings or marginalization. We refer the reader to [68, 132] for more details.

Definition 9.3.2. *Two POVMs* $A = (A_1, \ldots, A_k)$, $B = (B_1, \ldots, B_l)$ *on* $\mathcal{M}_d$ *are called* compatible *if there exists a* joint POVM $C = (C_{11}, \ldots, C_{kl})$ *on* $\mathcal{M}_d$ *such that* A *and* B *are its respective* marginals*:*

$$\forall i \in [k], \qquad A_i = \sum_{j=1}^{l} C_{ij}.$$

$$\forall j \in [l], \qquad B_j = \sum_{i=1}^{k} C_{ij}.$$

More generally, a g-tuple of POVMs $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)})$ is called compatible if there exists a POVM C with outcome set $[k_1] \times \cdots \times [k_g]$ such that, for all $x \in [g]$, the POVM $A^{(x)}$ is the x-th marginal of C:

$$\forall i_x \in [k_x], \qquad A^{(x)}_{i_x} = \sum_{i_1=1}^{k_1} \cdots \sum_{i_{x-1}=1}^{k_{x-1}} \sum_{i_{x+1}=1}^{k_{x+1}} \cdots \sum_{i_g=1}^{k_g} C_{i_1 i_2 \cdots i_g}$$
$$= \sum_{\substack{\mathbf{j} \in [k_1] \times \cdots \times [k_g] \\ j_x = i_x}} C_{\mathbf{j}}.$$

Note that the definition of compatibility given above can be formulated as a (feasibility) semi-definite program (SDP) [70]. One can equivalently formulate the notion of compatibility with more general post-processings.

Proposition 9.3.3. *An N-tuple of POVMs $\mathbf{A} = (A^{(1)}, \ldots, A^{(N)})$ is compatible if and only if there exists a joint POVM $(C_k)_{k\in[K]}$ and a family of conditional probabilities $\big(p_x(\cdot|\cdot)\big)_{x\in[N]}$ such that*

$$\forall x \in [N],\, \forall i \in [k_x], \qquad A_i^{(x)} = \sum_{k\in[K]} p_x(i|k)C_k.$$

We now consider the simplest possible setting, that of two 2-outcome POVMs $\{Q, I-Q\}$ and $\{P, I-P\}$, where P, Q are $d\times d$ self-adjoint matrices satisfying $O \leq P, Q \leq I_d$. The pair of POVMs is compatible if and only if $\varepsilon_0 \leq 1$ [9], where

$$\varepsilon_0 := \inf\big\{\varepsilon\,:\,\exists\delta\geq 0 \quad \text{s.t.} \quad \delta + I - Q - P \geq 0,\, Q + \varepsilon I - \delta \geq 0,\, P + \varepsilon I - \delta \geq 0\big\}, \tag{9.1}$$

where δ is a positive semidefinite matrix. The above formula corresponds to the value of a semidefinite program encoding the existence of a joint measurement for the POVMs $\{P, I-P\}$ and $\{Q, I-Q\}$. Generally, every SDP comes with a dual formulation. In our case the dual SDP is given below [9]:

Lemma 9.3.4. *Given the above optimization problem for deciding compatibility, its dual formulation is given by:*

$$\varepsilon^* = \sup_{X,Y,Z\geq 0}\Bigg\{\operatorname{Tr}[X(Q+P-I)] - \operatorname{Tr}[YQ] - \operatorname{Tr}[PZ] \textit{ with } X \leq Y+Z,\, \operatorname{Tr}[Y+Z] = 1\Bigg\},$$

Proof. Let us consider the following Lagrangian, corresponding the primal SDP (10.1).

$$\mathcal{L} := \varepsilon - \langle X, \delta + I - Q - P\rangle - \langle Y, \varepsilon I + Q - \delta\rangle - \langle Z, \varepsilon I + P - \delta\rangle - \langle C, \delta\rangle$$

Above X, Y, Z, C are positive semidefinite matrices which represent the constraints of the primal optimisation problem. Due to the strict feasibility of the SDP we can calculate its dual optimal value is the same as the optimal one of the primal (Slater's condition, see [70]). Thus, we have the following equality:

$$\inf_{\varepsilon,\delta}\sup_{X,Y,Z,C}\mathcal{L} = \sup_{X,Y,Z,C}\inf_{\varepsilon,\delta}\mathcal{L}.$$

A simple calculation shows that

$$\inf_{\varepsilon,\delta}\mathcal{L} = \langle X, Q+P-I\rangle - \langle Y, Q\rangle - \langle P, Z\rangle$$

with $\operatorname{Tr}[Y+Z] = 1$ and $Z + Y - X - C = 0 \iff X \leq Y + Z$, which is precisely the dual formulation from the statement. □

In the following, we shall use the SDP value to describe the compatibility threshold of the POVMs, that is the minimal quantity of noise that one needs to mix in, in order to render the POVMs compatible; such quantities go in the literature under the name of *robustness of incompatibility* [97].

Definition 9.3.5. *For a given parameter $\eta \in [0,1]$, given two POVMs $A = (A_1, \ldots, A_k)$, $B = (B_1, \ldots, B_l)$ on $\mathcal{M}_d$ one define their* noisy *version as $A^\eta := (A_1^\eta, \ldots, A_k^\eta)$, $B^\eta := (B_1^\eta, \ldots, B_l^\eta)$ with*

$$\forall i \in [k], \qquad A_i^\eta := \eta A_i + (1-\eta)\frac{I}{k}.$$

$$\forall j \in [l], \qquad B_j^\eta := \eta B_j + (1-\eta)\frac{I}{l}.$$

Remark 9.3.6. *In our simplified setting, we shall only consider POVMs with two outcomes $\mathcal{P} = \{P, I-P\}$, $\mathcal{Q} = \{Q, I-Q\}$ and their noisy versions. The definition given above can be rewritten as follows:*

$$\mathcal{P}^\eta = \{P^\eta, I - P^\eta\},$$

and

$$\mathcal{Q}^\eta = \{Q^\eta, I - Q^\eta\}.$$

The measurements $\mathcal{P}^\eta$ and $\mathcal{Q}^\eta$ can also be seen as convex mixtures of the measurement $\mathcal{P}$ and $\mathcal{Q}$ with the trivial POVM $\mathcal{I} = (\frac{I}{2}, \frac{I}{2})$. One has $\mathcal{Q}^\eta = \eta\mathcal{Q} + (1-\eta)\mathcal{I}$ and $\mathcal{P}^\eta = \eta\mathcal{P} + (1-\eta)\mathcal{I}$.

Let us now formalize the incompatibility robustness, in the symmetric case, where the same amount of white noise $(I/2, I/2)$ is mixed into the two POVMs; for the asymmetric version, see the incompatibility regions defined in [80, Section III].

Definition 9.3.7. *For two (binary) measurements $\mathcal{P}$, $\mathcal{Q}$, we define their* noise compatibility threshold *as:*

$$\Gamma(P, Q) := \sup\left\{\eta \in [0,1] \,:\, \mathcal{P}^\eta, \mathcal{Q}^\eta \text{ are compatible }\right\}.$$

Proposition 9.3.8. *The noise compatibility threshold for two (binary) measurements $\mathcal{P}$ and $\mathcal{Q}$ is given by:*

$$\Gamma(P, Q) = \frac{1}{1 + 2\varepsilon^*},$$

where ε^* is the optimal value of the SDP from Lemma 9.3.4.

Proof. Recalling that $\mathcal{P}^\eta, \mathcal{Q}^\eta$ are compatible is equivalent to $\exists\delta \geq 0$ with the following conditions:

$$\begin{aligned}\eta Q + (1-\eta)\frac{I}{2} - \delta &\geq 0\\ \eta P + (1-\eta)\frac{I}{2} - \delta &\geq 0\\ \delta - \eta(P + Q - I) &\geq 0\end{aligned}$$

Where it is easy to see that is equivalent to

$$\begin{aligned}Q + \varepsilon I - \delta' &\geq 0\\ P + \varepsilon I - \delta' &\geq 0\\ \delta' - (P + Q - I) &\geq 0\end{aligned}$$

with $\delta' := \frac{\delta}{\eta}$ and $\varepsilon = \frac{1}{2}(\frac{1}{\eta} - 1) \iff \eta = \frac{1}{2\varepsilon+1}$. By tacking the supremum over η to compute the noise compatibility threshold $\Gamma(P, Q)$ with the following constraints:

$$\begin{aligned}Q + \varepsilon I - \delta' &\geq 0\\ P + \varepsilon I - \delta' &\geq 0\\ \delta' - (P + Q - I) &\geq 0\end{aligned}$$

is given by

$$\begin{aligned}\Gamma(P, Q) &= \sup\left\{\frac{1}{2\varepsilon + 1}\,\middle|\, \exists\delta' \geq 0\,,\, Q + \varepsilon I - \delta' \geq 0,\, P + \varepsilon I - \delta' \geq 0,\, \delta' - (P + Q - I) \geq 0\right\}\\ &= \frac{1}{2\inf\left\{\varepsilon \,\middle|\, P, Q \text{ compatible}\right\} + 1} = \frac{1}{2\varepsilon_0 + 1} = \frac{1}{2\varepsilon^* + 1}\end{aligned}$$

which ends the proof of the proposition. □

9.4 Tensor product of Banach spaces

In this section we wil give a brief overview of tensor norms with the aim of presenting Bell inequalities in the tensor norm framework. Tensor norms provide the natural mathematical framework for Bell inequalities, see the following survey [126] and the reference therein. Let us start by recalling the projective and injective tensor norms for (finite-dimensional) Banach spaces.

Definition 9.4.1. *Given two finite-dimensional Banach spaces X and Y with their respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and $z \in X \otimes Y$, we define the* projective tensor norm *of z as:*

$$\|z\|_{X\otimes_\pi Y} := \inf\left\{\sum_{i=1}^N \|x_i\|_X \|y_i\|_Y : z = \sum_{i=1}^N x_i \otimes y_i\right\},$$

where the infimum is taken over all the decompositions of $z = \sum_{i=1}^N x_i \otimes y_i$ where N is a finite but arbitrary integer. We write $X \otimes_\pi Y = (X \otimes Y, \|\cdot\|_{X\otimes_\pi Y})$, the Banach space induced by the projective tensor norm on $X \otimes Y$.

Every Banach space comes with a dual:

Definition 9.4.2. *Let X a finite-dimensional Banach space. The space of all bounded linear functionals on X is called its dual space and denoted by X^*. It comes equipped with a norm:*

$$\forall \varphi \in X^*, \qquad \|\varphi\|_{X^*} := \sup_{\|x\|_X \leq 1} |\varphi(x)|.$$

We now introduce the other tensor norm of importance to us.

Definition 9.4.3. *Given two finite-dimensional Banach spaces X and Y with their respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and $z \in X \otimes Y$, we define the* injective tensor norm *of z as:*

$$\|z\|_{X\otimes_\varepsilon Y} := \sup_{\alpha\in\mathbb{B}(X^*),\beta\in\mathbb{B}(Y^*)} |\langle z, \alpha\otimes\beta\rangle|,$$

where $\mathbb{B}(X^*)$ and $\mathbb{B}(Y^*)$ are the unit balls of X^* and Y^*.

We write $X \otimes_\varepsilon Y = (X \otimes Y, \|\cdot\|_{X\otimes_\varepsilon Y})$, the Banach space induced by the injective norm on $X \otimes Y$.

It is known that the projective and the injective tensor product play the role of maximal and the minimal norm respectively that we can put naturally in the algebraic tensor product, for that we give the following definition of a reasonable crossnorm.

Definition 9.4.4. *Let $z \in X \otimes Y$, we say that a norm α on $X \otimes Y$ given by $\|z\|_{X\otimes_\alpha Y}$ is a* reasonable crossnorm *(or a* tensor norm*) if for $z = x \otimes y$ we have:*

$$\|z\|_{X\otimes_\alpha Y} \leq \|x\|_X \|y\|_Y$$

and the dual $\varphi = \varphi_1 \otimes \varphi_2 \in X^* \otimes Y^*$ satisfies

$$\|\varphi\|_{X^*\otimes_\alpha Y^*} \leq \|\varphi_1\|_{X^*} \|\varphi_2\|_{Y^*}.$$

We write $X \otimes_\alpha Y = (X \otimes Y, \|\cdot\|_{X\otimes_\alpha Y})$, the Banach space induced by α on $X \otimes Y$.

The definition above can be found in [33, page 127], with the following equivalent statement.

Proposition 9.4.5. *[33, Proposition 6.1] Consider two finite-dimensional Banach spaces X and Y with their respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. A norm α on $X \otimes Y$ is a reasonable crossnorm if and only if for all $z \in X \otimes Y$, we have*

$$\|z\|_{X\otimes_\varepsilon Y} \leq \|z\|_{X\otimes_\alpha Y} \leq \|z\|_{X\otimes_\pi Y}.$$

Remark 9.4.6. *The injective and the projective tensor norm are dual to each other, in the following sense [33]:*

$$\|z\|_{X\otimes_\pi Y} := \sup_{\|\alpha\|_{X^*\otimes_\varepsilon Y^*}\leq 1} \langle \alpha, z\rangle,$$
$$\|z\|_{X\otimes_\varepsilon Y} := \sup_{\|\alpha\|_{X^*\otimes_\pi Y^*}\leq 1} \langle \alpha, z\rangle.$$

In general, for each tensor norm $\|\cdot\|_{X\otimes_\alpha Y}$ we can define its *dual tensor norm* we note it by α^* and we have the following definition.

Definition 9.4.7. *Consider two finite dimensional Banach spaces X and Y with their respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Let $z\in X\otimes Y$ and $v\in X^*\otimes Y^*$, the dual tensor norm α^* of a given tensor norm α is defined by*

$$\|z\|_{X\otimes_{\alpha^*}Y} := \sup\{|\langle v, z\rangle|;\ \|v\|_{X^*\otimes_\alpha Y^*}\leq 1\}.$$

We write $X\otimes_{\alpha^*} Y = (X\otimes Y, \|\cdot\|_{X\otimes_{\alpha^*}Y})$, the Banach space induced by the norm α^* on $X\otimes Y$.

Remark 9.4.8. *With the definition above, we have the nice identification between the dual space of the tensor product of two spaces endowed with a tensor norm α and the dual space of each of the two spaces endowed with the dual norm α^* where we have*

$$(X\otimes_\alpha Y)^* = X^*\otimes_{\alpha^*} Y^*.$$

One last definition we want to recall that will play a fundamental role for Bell inequalities which is the reasonable norm known as γ_2 norm.

Definition 9.4.9. *Given two finite-dimensional Banach spaces X and Y with their respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, define the tensor norm γ_2 of $z\in X\otimes Y$ by:*

$$\|z\|_{X\otimes_{\gamma_2}Y} := \inf\left\{\sup_{\alpha^*\in\mathbb{B}(X^*)}\left(\sum_{i=1}^N|\alpha^*(x_i)|^2\right)^{\frac{1}{2}}\sup_{\beta^*\in\mathbb{B}(Y^*)}\left(\sum_{j=1}^N|\beta^*(y_j)|^2\right)^{\frac{1}{2}} : z=\sum_{i=1}^N x_i\otimes y_i\right\},$$

where the infimum is taken over all decompositions of $z=\sum_{i=1}^N x_i\otimes y_i$ with $x_i\in X$ and $y_j\in Y$.

9.5 Bell inequalities and non-local games

In this section we introduce the notion of *non-local games* and show how it incorporates the non-local properties of the quantum world via the well-known Bell inequalities. We then recall how the natural framework for these games is the metric theory of tensor products of finite-dimensional Banach spaces.

The non-local aspect of quantum mechanics can be incorporated in the non-local game framework. In this paper, we only consider non-local games with two players, *Alice* and *Bob*, which are collaborating to win the game. During the game, a third person known as the *Referee*, will ask a certain number of question to the players which are not allowed to communicate. The two protagonists reply cooperatively to the referee with some answers. The referee will decide to accept or reject the answers, declaring a win or a loss. Note that in this paper we are going to consider games with arbitrary number of questions N, but only with two answers ($+1$ or -1); in the general case, Alice and Bob can give answers from a fixed set of given cardinality.

During the game, players have access to a predetermined set of resources: this determines the type of strategy they are permitted to use. In this work, we shall consider *classical strategies* and *quantum strategies*.

In classical strategies, the players share samples from a classical random variable that they can use to produce their answers locally. When using quantum strategies, the players share a bipartite quantum state, on which they can act locally with transformations and measurements.

We shall focus on *correlation games*, where the payoff of the game depends on the correlation of the ± 1 answers ab, weighted by real numbers M_{xy} depending on the question:

$$\text{payoff} = \sum_{x,y \in [N]} \sum_{a,b \in \{\pm 1\}} M_{xy}\, ab \cdot \mathbb{P}(a,b|x,y),$$

where $\mathbb{P}(a,b|x,y)$ is the (strategy-dependent) probability that Alice and Bob answer respectively a and b, when presented with the questions $x, y \in [N]$. The matrix $M \in \mathcal{M}_N(\mathbb{R})$ encodes the rules of the game, and it is called the *Bell functional* [126]. In the following, we discuss the optimal classical and quantum strategies for a given non-local game M.

Definition 9.5.1. *The* classical bias of the game M *is defined as the optimisation problem*

$$\beta(M) := \sup \Big| \sum_{x,y=1}^{N} \sum_{a,b \in \{\pm 1\}} M_{xy}\, ab\, \mathbb{P}_c(a,b|x,y) \Big|$$

where the supremum is taken over all classical strategies

$$\mathbb{P}_c(a,b|x,y) = \int_\Lambda \mathbb{P}_A(a|x,\lambda)\, \mathbb{P}_B(b|y,\lambda)\, \mathrm{d}\mu(\lambda).$$

Above, $\mathbb{P}_A$, resp. $\mathbb{P}_B$ correspond to Alice's, resp. Bob's strategies, which can depend on the shared random variable λ having distribution μ.

Introducing the expectation values with respect to the outputs a, b

$$A_x(\lambda) := \sum_{a \in \{\pm 1\}} a\, \mathbb{P}_A(a|x,\lambda) \qquad \text{and} \qquad B_y(\lambda) := \sum_{b \in \{\pm 1\}} b\, \mathbb{P}_B(b|y,\lambda),$$

we have

$$\begin{aligned}
\beta(M) &= \sup_{\mathbb{P}_A, \mathbb{P}_B, \mu} \Big| \sum_{x,y=1}^{N} \sum_{a,b \in \{\pm 1\}} M_{xy}\, a\, b \int_\Lambda \mathbb{P}_A(a|x,\lambda)\, \mathbb{P}_B(b|y,\lambda)\, \mathrm{d}\mu(\lambda) \Big| \\
&= \sup_{A_x, B_y, \mu} \Big| \sum_{x,y=1}^{N} M_{xy} \int_\Lambda A_x(\lambda)\, B_y(\lambda)\, d\mu(\lambda) \Big| \\
&= \sup_{\gamma} \Big| \sum_{x,y=1}^{N} M_{xy}\, \gamma_{x,y} \Big|,
\end{aligned}$$

where the matrix $\gamma = (\gamma_{x,y})$ is a classical correlation matrix, containing the relevant information from the set of classical strategies.

Definition 9.5.2. *We define the set of* classical correlations *as*

$$\mathbb{L} := \left\{ \gamma_{x,y} \,\Big|\, \gamma_{x,y} = \int_\Lambda A_x(\lambda) B_y(\lambda)\, \mathrm{d}\mu(\lambda);\ |A_x(\lambda)|, |B_y(\lambda)| \leq 1 \right\} \subseteq \mathcal{M}_N(\mathbb{R})$$

where λ is a random variable shared by Alice and Bob, following a probability distribution μ.

Using the definition above, the maximum payoff of a game M, using classical strategies, can be understood as the maximum overlap of the Bell functional M defining the game with the set of classical correlations.

Proposition 9.5.3. *The classical bias of the game defined by a Bell functional M is:*

$$\beta(M) = \sup_{\gamma \in \mathbb{L}} \left\{ \Big| \sum_{x,y=1}^{N} M_{xy}\, \gamma_{x,y} \Big| \right\}.$$

We now move on to the quantum setting, where the players are allowed to use quantum strategies, that is they are allowed to perform local operations on a shared entangled state.

Definition 9.5.4. *The* quantum bias of the game M *is defined as the optimisation problem*

$$\beta^*(M) := \sup \Big| \sum_{x,y=1}^{N} \sum_{a,b \in \{\pm 1\}} M_{xy}\, a\, b\, \mathbb{P}_q(a,b|x,y) \Big|$$

where the supremum is taken over all quantum strategies

$$\mathbb{P}_q(a,b|x,y) = \operatorname{Tr}\left[\rho\,(A_{a|x} \otimes B_{b|y})\right],$$

where ρ is a bipartite shared quantum state (of arbitrary dimension), and, for all questions x, y, $(A_{\pm|x})$, resp. $(B_{\pm|y})$ are POVMs on Alice's, resp. Bob's quantum system.

Introducing the operators

$$A_x := \sum_{a \in \{\pm 1\}} a\, A_{a|x} \qquad \text{and} \qquad B_y := \sum_{b \in \{\pm 1\}} b\, B_{b|y},$$

and performing a similar computation as in the case of classical strategies, we are led to following definition and expression for the quantum bias of a non-local correlation game M.

Definition 9.5.5. *We define the set of* quantum correlations *as*

$$\mathbb{Q} := \left\{ \gamma_{x,y} \,\Big|\, \gamma_{x,y} = \operatorname{Tr}\left[\rho \cdot (A_x \otimes B_y)\right];\ \|A_x\|_\infty, \|B_y\|_\infty \leq 1 \right\} \subseteq \mathcal{M}_N(\mathbb{R}).$$

Above, ρ is a bipartite quantum state of arbitrary dimension, and A_x, B_y are observables of norm less than one.

Proposition 9.5.6. *The quantum bias of the game defined by a Bell functional M is:*

$$\beta^*(M) = \sup_{\gamma \in \mathbb{Q}} \left\{ \Big| \sum_{x,y=1}^{N} M_{xy}\, \gamma_{x,y} \Big| \right\}.$$

Remark 9.5.7. *The correlation games discussed above are also known in the literature as* XOR games*, when the set of outputs is $\{0,1\}$ (instead of $\{\pm 1\}$) [133].*

Since classical correlations are a subset of the quantum correlations (corresponding to diagonal operators A_x, B_y), the quantum bias of the game must be always larger or equal the classical bias. In some cases, the quantum bias $\beta^*(M)$ is strictly larger than the classical one, which can be understood physically as the existence of quantum correlations can not be reproduced within a classical local hidden variable model. This motivates the following definition.

Definition 9.5.8. *For a given non-local game described by a* Bell functional M*, we say that we have a* Bell violation *if $\beta^*(M) > \beta(M)$.*

Now we will recall the results on the profound link between the classical bias of a game and its quantum bias within their respective tensor norm description.

Theorem 9.5.9. *[126] Consider a non-local correlation game characterized by the matrix $M \in \mathcal{M}_N(\mathbb{R})$.*

- *The classical bias of the game is equal to the injective tensor norm of M:*

$$\beta(M) = \|M\|_{\ell_1^N(\mathbb{R}) \otimes_\varepsilon \ell_1^N(\mathbb{R})}.$$

- *The quantum bias of the game is equal to the γ_2^* tensor norm of M:*

$$\beta^*(M) = \|M\|_{\ell_1^N(\mathbb{R}) \otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})}.$$

where we recall from the Definition 9.4.7 that

$$\|M\|_{\ell_1^N(\mathbb{R}) \otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})} := \sup\Big\{ |\langle v, M\rangle| \,;\, \|v\|_{\ell_\infty^N(\mathbb{R}) \otimes_{\gamma_2} \ell_\infty^N(\mathbb{R})} \le 1 \Big\}.$$

Tsirelson showed in [134] the following theorem that links the classical and the quantum bias of an XOR game with famous Grothendieck constant $K_G^{\mathbb{R}}$ that plays a fundamental role in the theory of tensor product of Banach spaces, see also [126, Corollary 3.3].

Theorem 9.5.10. *Consider a non-local correlation game characterized by the matrix $M \in \mathcal{M}_N(\mathbb{R})$.*

$$\beta^*(M) \le K_G^{\mathbb{R}} \beta(M).$$

From the result above, one can easily see that Bell inequality violations ($\beta^*(M) > \beta(M)$) can be understood as tensor norm ratios. The result above shows the intrinsic link between Bell inequality violation and tensor norms, which motivates our framework on using tensor norms.

Let us now discuss the CHSH non-local game [127].

Definition 9.5.11. *The CHSH game is given by the particular* Bell functional *defined as the following:*

$$M_{CHSH} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

In this subsection, we recall the result of [9] where they made the link between the maximal violation of Bell inequality and the compatibility of the quantum measurement in the CHSH game. Precisely the maximal violation of the CHSH inequality is equivalent to the dual of formulation of the compatibility problem as an SDP [9].

Theorem 9.5.12. *Two dichotomic measurements $A = (A_0, A_1)$, $B = (B_0, B_1)$ are incompatible if and only if they enable violation of the CHSH inequality. More precisely, the optimal value of the CHSH inequality is related to*

$$\sup_{\psi, B_0, B_1} \langle\psi|\, \mathbb{B} \,|\psi\rangle = \frac{1}{\Gamma(A)}.$$

with

$$\mathbb{B} := \sum_{x,y=0}^{1} M_{CHSH}(x,y) A_x \otimes B_y = \frac{1}{2}(A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1).$$

and $\Gamma(A)$ is the noise compatibility threshold.

Proof. The proof of this theorem is basically the following remark, where in [9] they have showed that

$$\sup_{\psi, B_0, B_1} \langle\psi|\, \mathbb{B} \,|\psi\rangle = 1 + 2\varepsilon^*.$$

By combining this result and the proposition 9.3.8 where Alice measurement apparatus are described by the POVMs $\{Q, I-Q\}$ and $\{P, I-P\}$ which ends the proof. □

9.6 The tensor norm associated to a game

In this section we will introduce the notion of nonlocality using our framework of tensor norms. For that we consider as the previous section a fixed *quantum game*, and for *fixed Alice measurements* we introduce the *M-Bell-(non)locality* notion. This quantity will characterise all the observed non-local effects in Alice side. To do so she will calculate the following tensor norm $\|A\|_M$ given by her fixed measurement apparatus. This quantity is obtained by optimizing over all the shared quantum state and all Bob measurement apparatus. We say that Alice measurement apparatus are *M-Bell-local* if such $\|A\|_M$ is *less than or equal the classical bias of the game* and if is not we say that her measurement are *M-Bell non-local*.

The physical motivation of such statement can be understood as the following, for a fixed quantum game no matter the optimisation over all the shared quantum states and Bob measurement we cannot do better than the classical bias of the game, which means that one cannot do better than the classical setting even if we use the quantum strategies.
For that we will give the precise definition of $\|A\|_M$ and the *M-Bell-(non)locality* notion. We will show the main theorem of this section that $\|A\|_M$ is a *tensor norm* in $(\mathbb{R}^N, \|\cdot\|_M) \otimes (\mathcal{M}_d, \|\cdot\|_\infty)$ for a fixed invertible quantum game M.

As a starting point, we give the two main definitions of this section.

Definition 9.6.1. *Consider a fixed N-input, 2-outcome non-local game $M \in \mathcal{M}_N(\mathbb{R})$. Fix also Alice's measurements, a N-tuple of binary observables $A = (A_1, \ldots, A_N) \in \mathcal{M}_d^{sa}(\mathbb{C})^N$. The largest quantum bias of the game M, with Alice using the observable A_x to answer question $x \in [N]$, is given by*

$$\sup_{\|\psi\|=1} \sup_{\|B_y\|\leq 1} \left\langle \psi \middle| \sum_{x,y=1}^{N} M_{xy}\, A_x \otimes B_y \middle| \psi \right\rangle = \sup_{\|B_y\|\leq 1} \lambda_{\max} \left[\sum_{x,y}^{N} M_{xy}\, A_x \otimes B_y \right] =: \|A\|_M,$$

where the suprema are taken over bipartite pure states $\psi \in \mathbb{C}^d \otimes \mathbb{C}^D$ and over Bob's observables $B = (B_1, \ldots, B_N) \in \mathcal{M}_D^{sa}(\mathbb{C})^N$, where D is a free dimension parameter. We shall later show in Theorem 9.6.10 that this quantity defines a (tensor) norm.

Remark 9.6.2. *In the definition above, the dimension of Alice's measurements is fixed (d), while the dimension of Bob's Hilbert space (D) is free. In the following we will show that one can assume, without loss of generality, that Alice and Bob have Hilbert spaces of the same dimension ($D = d$ suffices in the optimization problem).*

Let us consider $D \geq d$, a quantum state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^D$, and N binary measurement operators $B_1, \ldots, B_N \in \mathcal{M}_D^{sa}(\mathbb{C})$. The idea is that the Schmidt decomposition of the bipartite pure quantum state $|\psi\rangle$ will induce a reduction of the effective dimension of Bob's Hilbert space from D to d. We start from the Schmidt decomposition of $|\psi\rangle$

$$|\psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i}\, |a_i\rangle \otimes |b_i\rangle .$$

Note that in the equation above, the number of terms is bounded by the smallest of the two dimensions, that is d. The orthonormal family $\{|b_i\rangle\}_{i\in[d]}$ spans a subspace of dimension d inside $\mathbb{C}^D$. Consider an arbitrary orthonormal basis *$\{|\tilde{b}_i\rangle\}_{i\in[d]}$ of $\mathbb{C}^d$ and the isometry*

$$V : \mathbb{C}^d \to \mathbb{C}^D \qquad \textit{such that} \qquad \forall i \in [d], \quad V\,|\tilde{b}_i\rangle = |b_i\rangle .$$

Let us now introduce the quantum state

$$\mathbb{C}^d \otimes \mathbb{C}^d \ni |\tilde{\psi}\rangle := \sum_{i=1}^{d} \sqrt{\lambda_i}\, |a_i\rangle \otimes |\tilde{b}_i\rangle$$

and the measurement operators

$$\mathcal{M}_d^{sa}(\mathbb{C}) \ni \tilde{B}_y := V^* B_y V, \quad \forall y \in [N].$$

The normalization of the state and the fact that the $\tilde{B}_y$ are contractions follow from the isometry property of the operator V. We now have

$$\begin{aligned}
\Big\langle \psi \Big| \sum_{x,y=1}^N M_{xy} A_x \otimes B_y \Big| \psi \Big\rangle &= \sum_{x,y=1}^N M_{xy} \sum_{i,j=1}^d \sqrt{\lambda_i \lambda_j} \langle a_i | A_x | a_j \rangle \underbrace{\langle b_i | B_y | b_j \rangle}_{=\langle \tilde{b}_i | V^* B_y V | \tilde{b}_j \rangle} \\
&= \sum_{x,y=1}^N M_{xy} \sum_{i,j=1}^d \sqrt{\lambda_i \lambda_j} \langle a_i | A_x | a_j \rangle \langle \tilde{b}_i | \tilde{B}_y | \tilde{b}_j \rangle \\
&= \Big\langle \tilde{\psi} \Big| \sum_{x,y=1}^N M_{xy} A_x \otimes \tilde{B}_y \Big| \tilde{\psi} \Big\rangle.
\end{aligned}$$

The above computation shows that any correlation that can be obtained with Bob's Hilbert space of dimension D can also be obtain with a Hilbert space of dimension d, equal to that of Alice.

Definition 9.6.3. *Given a non-local game M, we say that Alice's measurements $A = (A_1, \ldots, A_N)$ are M-*Bell-local *if for any choice of Bob's observables B and for any shared state ψ, one cannot violate the Bell inequality corresponding to M:*

$$\|A\|_M \leq \beta(M).$$

*If this is not the case, we call Alice's measurements M-*Bell-non-local.

Instead of using definition 9.6.1 we will use another simple equivalent formulation of $\|A\|_M$. To do so, we will consider $\|A\|_M$ as an optimization problem using an SDP, and we will give its equivalent formulation as a dual of the primal SDP.

Lemma 9.6.4. *Given a quantum game $(M_{xy})_{\{x,y=1\}}^N$ we can characterise the following equivalent formulation of $\|A\|_M$:*

$$\|A\|_M = \lambda_{\max} \left[\sum_{y=1}^N \left| \sum_{x=1}^N M_{xy} \, A_x \right| \right].$$

Proof. Remark that the definition above is equivalent to

$$\|A\|_M = \sup_{\|\psi\|=1} \quad \sup_{\|B_y\| \leq 1} \Big\langle \psi \Big| \sum_{x,y=1}^N M_{xy} \, A_x \otimes B_y \Big| \psi \Big\rangle$$

with $|\psi\rangle = \sqrt{\rho} \otimes I \sum_{i=1}^d |ii\rangle$ and ρ is a density matrix (this is the classical purification trick in quantum information theory, expressing any bipartite pure state as a local perturbation of the maximally entangled state $\sum_i |ii\rangle$). Then, one has

$$\|A\|_M = \sup_{\rho \geq 0 \, ; \, \mathrm{Tr}\, \rho = 1} \quad \sup_{\|B_y\| \leq 1} \left\{ \sum_{x,y=1}^N M_{xy} \, \mathrm{Tr} \left[\sqrt{\rho} \, A_x \, \sqrt{\rho} \, B_y^\top \right] \right\}$$

Using the following variable change $B_y^\top = 2S_y - I$ for all y, we have

$$\|A\|_M = \sup_{S_y \, , \, \rho} \left\{ \sum_{y=1}^N \mathrm{Tr} \left[\sqrt{\rho} \, A'_y \, \sqrt{\rho} \, (2S_y - I) \right] \right\}$$

where $A'_y = \sum_{x=1}^N M_{xy}\, A_x$ and the optimisation problem is on S_y for all y and ρ with the following constraint : $0 \le S_y \le I$ and $\rho \ge 0$ and $\mathrm{Tr}\,\rho = 1$.

Then

$$\|A\|_M = \sup_{S_y\,,\,\rho}\left\{\sum_{y=1}^N \mathrm{Tr}\left[A'_y(2\sqrt{\rho}\,S_y\,\sqrt{\rho}-\rho)\right] \,:\, 0 \le S_y \le I\,,\, \rho \ge 0\,,\, \mathrm{Tr}\,\rho = 1\right\}$$

We consider now one last change of variable $S'_y = \sqrt{\rho}\,S_y\,\sqrt{\rho}$.

$$\|A\|_M = \sup_{S'_y\,,\,\rho}\left\{\sum_{y=1}^N \mathrm{Tr}\left[A'_y(2S'_y-\rho)\right] \,:\, 0 \le S'_y \le \rho\,,\, \rho \ge 0\,,\, \mathrm{Tr}\,\rho = 1\right\}$$

where the optimisation problem now is on S'_y for all y and ρ with the constrained above.

We will formulate $\|A\|_M$ as an SDP and we will compute its dual.

For that, we consider the following Lagrangian :

$$\mathcal{L} = \sum_{y=1}^N \mathrm{Tr}\left[A'_y\,(2S'_y-\rho)\right] + \langle X,\rho\rangle + \sum_{y=1}^N \langle X_y, S'_y\rangle + \varepsilon(1-\mathrm{Tr}\,\rho) + \sum_{y=1}^N \langle \rho - S'_y, Z_y\rangle.$$

with X, X_y, ε, Z_y are the constraints respectively for $\rho \ge 0$, $S'_y \ge 0$, $\mathrm{Tr}\,\rho = 1$ and $S'_y \le \rho$. Then by using the SDP duality one has

$$\|A\|_M = \sup_{S'_y\,,\,\rho}\ \inf_{X\,,\,X_y\,,\varepsilon\,,Z_y} \mathcal{L} = \inf_{X\,,\,X_y\,,\varepsilon\,,Z_y}\ \sup_{S'_y\,,\,\rho} \mathcal{L}$$

with the following constraints:

- $X \ge 0$ and $\forall y$ $X_y\,, Z_y \ge 0$ are positive semidefinite matrices
- $\varepsilon \in \mathbb{R}$ is unconstrained

Using the duality given above and by tacking the suprema first over S'_y and ρ, we have:

$$\sup_{S'_y\,,\rho} \mathcal{L} = \begin{cases} \varepsilon\,, & \forall y\,,\, 2A'_y + X_y - Z_y = 0 \quad\text{and}\quad X - \sum_{y=1}^N A'_y - \varepsilon I + \sum_{y=1}^N Z_y = 0. \\ +\infty & \end{cases}$$

Now by tacking the infimum over the constraints

$$\begin{aligned}\|A\|_M &= \inf_{X\,,\,X_y\,,\,\varepsilon\,,\,Z_y}\left\{\varepsilon\,\Big|\ \ \forall y\,,\, 2A'_y + X_y - Z_y = 0\,;\, X - \sum_{y=1}^N A'_y - \varepsilon I + \sum_{y=1}^N Z_y = 0\right\} \\ &= \inf_{X_y\,,\,\varepsilon}\left\{\varepsilon\,\Big|\ \ \forall y\,,\, 2A'_y \le Z_y \quad\text{and}\quad \sum_{y=1}^N A'_y + \varepsilon I \ge \sum_{y=1}^N Z_y\right\}\end{aligned}$$

where in the last equality we have used that $X \ge 0$, $Z_y \ge 0$. With the constraints on Z_y and $Z_y \ge 2A'_y$, we can choose $Z_y := 2(A')^+_y$ with $(A')^+_y$ is the positive part of $A'_y = (A')^+_y - (A')^-_y$; this is the smallest (with respect to the positive semidefinite order) choice for Z_y. Using the optimal value above

$$\|A\|_M = \inf_{\varepsilon}\left\{\varepsilon\,\Big|\,\varepsilon I \ge \sum_{y=1}^N (A')^+_y + (A')^-_y\right\}$$

Then

$$\|A\|_M = \lambda_{\max}\left[\sum_{y=1}^N \left|\sum_{x=1}^N M_{xy}\,A_x\right|\right]$$

□

In the following, we shall exploit the new formulation of $\|A\|_M$ and we will show in the lemma below that $\|\cdot\|_M$ is a norm for any *invertible* game M.

Lemma 9.6.5. *Given an* invertible *game M, the M-Bell-locality quantity $\|A\|_M$ verifies the following two properties:*

$$\|A\|_M \geq 0,$$
$$\|A + A'\|_M \leq \|A\|_M + \|A'\|_M.$$

In particular, $\|\cdot\|_M$ is a norm.

Proof. To prove the first property we shall prove that :

- $\forall \alpha \in \mathbb{R}$ we have $\|\alpha A\|_M = |\alpha| \|A\|_M$.
- $\|A\|_M = 0 \implies A = 0$

Obviously we have

$$\|\alpha A\|_M = \lambda_{\max}\left[\sum_{y=1}^N \left|\sum_{x=1}^N \alpha M_{xy}\, A_x\right|\right] = |\alpha|\lambda_{\max}\left[\sum_{y=1}^N \left|\sum_{x=1}^N M_{xy}\, A_x\right|\right] = |\alpha|\|A\|_M$$

For the second property we have

$$\|A\|_M = \lambda_{\max}\left[\sum_{y=1}^N \left|\sum_{x=1}^N M_{xy}\, A_x\right|\right] = 0 \implies \sum_{x=1}^N M_{xy}\, A_x = 0 \implies A = 0$$

where the first implication is due to the positivity of $|M^\top A|$ and the second implication is obtained by the assumption of the invertibility of M and thus $M^\top$.
For the second property we have

$$\begin{aligned}\|A + A'\|_M &= \sup_{\|\psi\|=1} \sup_{\|B_y\|\leq 1} \left\langle \psi \right| \sum_{x,y=1}^N M_{xy}\,(A_x + A'_x) \otimes B_y \left| \psi \right\rangle \\ &\leq \sup_{\|\psi\|=1} \sup_{\|B_y\|\leq 1} \left\langle \psi \right| \sum_{x,y=1}^N M_{xy}\, A_x \otimes B_y \left| \psi \right\rangle + \sup_{\|\psi\|=1} \sup_{\|B_y\|\leq 1} \left\langle \psi \right| \sum_{x,y=1}^N M_{xy}\, A'_x \otimes B_y \left| \psi \right\rangle\end{aligned}$$

Hence we have

$$\|A + A'\|_M \leq \|A\|_M + \|A'\|_M.$$

□

In the last lemma we have shown that $\|A\|_M$ is a norm (for an *invertible Bell functional* M). We shall call this norm the *M-Bell-locality norm*.

The (real) vector spaces $\mathbb{R}^N$, resp. $\mathcal{M}_d^{sa}(\mathbb{C})$ shall be endowed with the $\|\cdot\|_M$, resp. the operator norm (or the Schatten-∞ norm, $\mathcal{S}_\infty$). Note that there is an abuse of notation here: we shall use $\|\cdot\|_M$ to denote norms on $\mathbb{R}^N$ and on $\mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$; the situation will be clear from the context. We shall now investigate the properties of the $\|\cdot\|_M$ norm with respect to this tensor product structure. We will consider that for given N-tuple of observables $(A_1, A_2, \ldots, A_N)$, we associate the tensor

$$A := \sum_{x=1}^N e_x \otimes A_x \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C}).$$

Definition 9.6.6. *Given $p \in \mathbb{R}^N$, we define the following quantity:*

$$\|p\|_M := \sum_{y=1}^{N} \Big| \sum_{x=1}^{N} M_{xy}\, p_x \Big| = \|M^\top p\|_1.$$

In the lemma below we will show that $\|\cdot\|_M$ is a norm.

Lemma 9.6.7. *Given an invertible matrix M, the function $\mathbb{R}^N \ni p \mapsto \|p\|_M$ is a norm.*

Proof. Obviously we have $\|\alpha\, p\|_M = |\alpha|\, \|p\|_M$ for all $\alpha \in \mathbb{R}$.
Now we will show that $\|p\|_M = 0 \implies p = 0$

$$\|p\|_M = \sum_{y=1}^{N} \Big| \sum_{x=1}^{N} M_{xy}\, p_x \Big| = 0 \implies \sum_{x=1}^{N} M_{xy}\, p_x = 0 \iff (M^\top p)_y = 0$$

by using the assumption that M is invertible we have necessarily $p = 0$, which ends the proof of $\|p\|_M \geq 0$.
Now we prove the triangle inequality $\|p + p'\|_M \leq \|p\|_M + \|p'\|_M$.
Let's consider

$$\|p+p'\|_M = \sum_{y=1}^{N} \Big| \sum_{x=1}^{N} M_{xy}\,(p_x + p'_x) \Big| \leq \sum_{y=1}^{N} \Big| \sum_{x=1}^{N} M_{xy}\, p_x \Big| + \sum_{y=1}^{N} \Big| \sum_{x=1}^{N} M_{xy}\, p'_x \Big| = \|p\|_M + \|p'\|_M$$

Thus we have shown that $\|\cdot\|_M$ is a norm. □

By the Lemma 9.6.7, we endow $\mathbb{R}^N$ with the norm $\|\cdot\|_M$, obtaining a Banach space $(\mathbb{R}^N, \|\cdot\|_M)$. In the following, we shall investigate the dual space of $(\mathbb{R}^N, \|\cdot\|_M)$. For that we shall compute the dual norm of $\|\cdot\|_M$ denoted by $\|\cdot\|_M^*$.

Proposition 9.6.8. *The dual norm $\|\cdot\|_M^*$ is given by:*

$$\forall p \in \mathbb{R}^N, \qquad \|p\|_M^* = \max_y \Big| \sum_{z=1}^{N} (M^{-1})_{yz}\, p_z \Big| = \|M^{-1} p\|_\infty.$$

Proof. Let $q, p \in \mathbb{R}^N$ we have

$$\begin{aligned}
|\langle p, q\rangle| &= \Big| \sum_{x=1}^{N} p_x\, q_x \Big| = \Big| \sum_{x,y,z=1}^{N} q_x\, M_{xy}\, M^{-1}_{yz} p_z \Big| = \Big| \sum_{y=1}^{N} \Big(\sum_{x=1}^{N} M_{xy}\, q_y \Big) \Big(\sum_{z=1}^{N} M^{-1}_{yz} p_z \Big) \Big| \\
&\leq \sum_{y=1}^{N} \Big| \sum_{x=1}^{N} M_{xy}\, q_x \Big| \Big| \sum_{z=1}^{N} M^{-1}_{yz}\, p_z \Big| \leq \Big(\max_y \Big| \sum_{z=1}^{N} M^{-1}_{yz}\, p_z \Big| \Big) \sum_{y=1}^{N} \Big| \sum_{x=1}^{N} M_{xy} q_x \Big| \\
&= \max_y \Big| \sum_{z=1}^{N} M^{-1}_{yz}\, p_z \Big| \, \|q\|_M.
\end{aligned}$$

where we have used in the second equality that $M \cdot M^{-1} = I$. By taking the supremum over $\|q\|_M \leq 1$, we have shown that $\|p\|_M^* \leq \|M^{-1}p\|_\infty$. To show the converse inequality, note that

$$\max_y \Big| \sum_{z=1}^{N} M^{-1}_{yz}\, p_z \Big| = \langle p, q\rangle, \qquad \text{for} \qquad q_z = \varepsilon (M^{-1})_{y_0 z}$$

for some $y_0 \in [d]$ achieving the maximum, and $\varepsilon = \pm 1$. In order to conclude, we have to establish that $\|q\|_M \leq 1$. Indeed, we have

$$\|q\|_M = \sum_y \Big| \sum_x M_{xy} \varepsilon (M^{-1})_{y_0 x} \Big| = \sum_y \big| (M^{-1}M)_{y_0 y} \big| = 1.$$

□

The Banach space $(\mathbb{R}^N, \|\cdot\|_M^*)$ is the dual of $(\mathbb{R}^N, \|\cdot\|_M)$: $(\mathbb{R}^N, \|\cdot\|_M)^* = (\mathbb{R}^N, \|\cdot\|_M^*)$. Now, we are ready to show the main theorem of this section, that the norm $\mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C}) \ni A \to \|A\|_M$ is a tensor norm (or a reasonable crossnorm) in the sense of Definition 9.4.4. To this end, using Proposition 9.4.5, it suffices to show that:

$$\|A\|_{\mathbb{R}^N \otimes_\varepsilon \mathcal{M}_d^{sa}(\mathbb{C})} \leq \|A\|_M \leq \|A\|_{\mathbb{R}^N \otimes_\pi \mathcal{M}_d^{sa}(\mathbb{C})}$$

where $\mathbb{R}^N$ is endowed with the norm $\|\cdot\|_M$ and $\mathcal{M}_d^{sa}$ with the norm $\|\cdot\|_\infty$. Before we show that $\|A\|_M$ is a tensor norm, we shall show the following proposition for tensors of rank one $A = p \otimes B \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}$.

Proposition 9.6.9. *Given $A \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$ with $\mathbb{R}^N$ and $M_d^{sa}(\mathbb{C})$ are endowed with $\|\cdot\|_M$ and the natural operator norm respectively. Given the particular decomposition $A = p \otimes B$ with $p \in (\mathbb{R}^N, \|\cdot\|_M)$ and $B \in (\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$, one has*

$$\|p \otimes B\|_M = \|p\|_M \|B\|_\infty.$$

Proof. Given $A = p \otimes B$ one has

$$\|p \otimes B\|_M = \lambda_{\max}\left[\sum_{y=1}^N \left|\sum_{x=1}^N M_{xy}\, p_x B\right|\right] = \lambda_{\max}[|B|] \sum_{y=1}^N \left|\sum_{x=1}^N M_{xy}\, p_x\right| = \|B\|_\infty \|p\|_M.$$

Above, we have used the following fact: for selfadjoint matrices B,

$$\|B\|_\infty = \max_{\lambda \text{ eig. of } B} |\lambda| = \max_{\lambda \text{ eig. of } |B|} \lambda = \lambda_{\max}[|B|].$$

□

We now state and prove the following important result, establishing that the norm $\|\cdot\|_M$ is indeed a tensor norm.

Theorem 9.6.10. *For a fixed N-input, 2-output invertible non-local game M, the quantity $\|\cdot\|_M$ introduced in Definition 9.6.1, which characterizes the largest quantum bias of the game M when one fixes Alice's dichotomic measurements, is a reasonable crossnorm on $\mathcal{M}_d^{sa}(\mathbb{C})^N \cong \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$:*

$$\|A\|_{\mathbb{R}^N \otimes_\varepsilon \mathcal{M}_d^{sa}(\mathbb{C})} \leq \|A\|_M \leq \|A\|_{\mathbb{R}^N \otimes_\pi \mathcal{M}_d^{sa}(\mathbb{C})}$$

with $(\mathbb{R}^N, \|\cdot\|_M)$ and $(\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$.

Before we give the proof of the theorem we recall the definitions of the projective and the injective norms in our setting:

$$\|A\|_{\mathbb{R}^N \otimes_\pi \mathcal{M}_d^{sa}(\mathbb{C})} := \inf\left\{\sum_{i=1}^k \|p_i\|_M \, \|X_i\|_\infty,\; A = \sum_{i=1}^k p_i \otimes X_i\right\}.$$

$$\|A\|_{\mathbb{R}^N \otimes_\varepsilon \mathcal{M}_d^{sa}(\mathbb{C})} := \sup\left\{\langle \pi \otimes \alpha, A\rangle;\; \|\pi\|_M^* \leq 1,\; \|\alpha\|_1 \leq 1\right\}.$$

with $\mathcal{M}_d^{sa}(\mathbb{C}) \ni \alpha \to \|\alpha\|_1 = \operatorname{Tr}|\alpha|$ is the Schatten 1-norm (or the nuclear norm).

Proof. We shall prove first the easy direction: $\|A\|_M \leq \|A\|_{\mathbb{R}^N \otimes_\pi \mathcal{M}_d^{sa}(\mathbb{C})}$. Let us consider a decomposition $A = \sum_{i=1}^k p_i \otimes X_i$. We have

$$\|A\|_M = \left\|\sum_{i=1}^k p_i \otimes X_i\right\|_M \leq \sum_{i=1}^k \|p_i \otimes X_i\|_M = \sum_{i=1}^k \|p_i\|_M \|X_i\|_\infty,$$

where the factorization property follows by Proposition 9.6.9. Hence we have

$$\|A\|_M \leq \|A\|_{\mathbb{R}^N \otimes_\pi \mathcal{M}_d^{sa}(\mathbb{C})}.$$

We shall now prove that $\|A\|_{\mathbb{R}^N \otimes_\varepsilon \mathcal{M}_d^{sa}(\mathbb{C})} \leq \|A\|_M$. Let $\alpha = \pm|\varphi\rangle\langle\varphi| \in (\mathcal{M}_d^{sa}$ be an extremal point of the unit ball of the $\mathcal{S}_1$ space and $\pi \in (\mathbb{R}^N, \|\cdot\|_M^*)$. We have

$$\begin{aligned}
|\langle \pi \otimes \alpha, A\rangle| &= \left|\left\langle \alpha, \sum_{x=1}^N \pi_x A_x \right\rangle\right| = \left|\left\langle \alpha, \sum_{x,y,z=1}^N A_z M_{zy} M_{yx}^{-1} \pi_x \right\rangle\right| \\
&= \sum_{y=1}^N \left|\sum_{z=1}^N M_{zy} \langle \alpha, A_z\rangle\right| \left|\sum_{x=1} M_{yx}^{-1} \pi_x\right| \leq \sum_{y=1}^N \left|\sum_{z=1}^N M_{zy} \langle \alpha, A_z\rangle\right| \max_y \left|\sum_{x=1} M_{yx}^{-1} \pi_x\right| \\
&= \|\pi\|_M^* \sum_{y=1}^N \left|\sum_{z=1}^N M_{zy} \langle \alpha, A_z\rangle\right| = \|\pi\|_M^* \sum_{y=1}^N \left|\sum_{z=1}^N \langle \alpha, M_{zy} A_z\rangle\right| \\
&= \|\pi\|_M^* \sum_{y=1}^N \left|\sum_{z=1}^N \mathrm{Tr}\left[\alpha M_{zy} A_z\right]\right| = \|\pi\|_M^* \sum_{y=1}^N \left|\sum_{z=1}^N \langle\varphi| \, M_{zy} A_z \, |\varphi\rangle\right| \\
&= \|\pi\|_M^* \sum_{y=1}^N \left| \langle\varphi| \left(\sum_{z=1}^N M_{zy} A_z\right)^+ |\varphi\rangle - \langle\varphi| \left(\sum_{z=1}^N M_{zy} A_z\right)^- |\varphi\rangle \right| \\
&\leq \|\pi\|_M^* \sum_{y=1}^N \left[\left| \langle\varphi| \left(\sum_{z=1}^N M_{zy} A_z\right)^+ |\varphi\rangle \right| + \left| \langle\varphi| \left(\sum_{z=1}^N M_{zy} A_z\right)^- |\varphi\rangle \right| \right] \\
&= \|\pi\|_M^* \sum_{y=1}^N \langle\varphi| \left|\sum_{z=1}^N M_{zy} A_z\right| |\varphi\rangle .
\end{aligned}$$

By taking the supremum $\|\pi\|_M^* \leq 1$ and $\|\alpha\|_{\mathcal{S}_1} \leq 1$ on the last expression we have:

$$\sup\{|\langle \pi \otimes \alpha, A\rangle|;\ \|\pi\|_M^* \leq 1\,,\ \|\alpha\|_{\mathcal{S}_1} \leq 1\} \leq \sup_{\|\varphi\|=1} \sum_{y=1}^N \langle\varphi| \left|\sum_{z=1}^N M_{zy} A_z\right| |\varphi\rangle = \lambda_{\max}\left[\sum_{y=1}^N \left|\sum_{z=1}^N M_{zy} A_z\right|\right].$$

Hence we have

$$\|A\|_{\mathbb{R}^N \otimes_\varepsilon \mathcal{M}_d^{sa}(\mathbb{C})} \leq \|A\|_M$$

which ends the proof of the theorem. □

9.7 Dichotomic measurement compatibility via tensor norms

Having addressed in the previous sections the maximum value of a non-local game M with fixed dichotomic observables on Alice's side A, we now turn to the second object of our study, quantum measurement (in-)compatibility. We characterize compatibility of dichotomic measurements (or quantum) using tensor norms, following [86]. Recall that we associate a dichotomic POVM $(E, I-E)$ to the corresponding observable $A = E-(I-E) = 2E-I$. In other words, the effect E corresponds to the "+1" outcome, while the effect $I-E$ corresponds to the other outcome, "−1". This way, the set of dichotomic POVMs is mapped to the set of selfadjoint operators $-I \leq A \leq I$.

To a N-tuple of observables $(A_1, A_2, \ldots, A_N)$, we associate the tensor

$$A := \sum_{i=1}^N e_i \otimes A_i \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C}). \tag{9.2}$$

The (real) vector spaces $\mathbb{R}^N$, resp. $\mathcal{M}_d^{sa}(\mathbb{C})$ shall be endowed with the ℓ_∞, resp. the operator norm (or the Schatten-∞ norm, S_∞). On the tensor product space

$$\mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C}) \cong [\mathcal{M}_d^{sa}(\mathbb{C})]^N$$

we shall consider two tensor norms: the *injective norm*

$$\|X = (X_1, X_2, \ldots, X_N)\|_\varepsilon = \max_{i=1}^{N} \|X_i\|_\infty \tag{9.3}$$

and the compatibility norm, which was introduced in [86, Proposition 9.4]. We review next its definition and its basic properties, in order to make the presentation self-contained. We note however that the situation considered in [86, Section 9] is more general, going beyond the case of quantum mechanics.

Definition 9.7.1. *For a tensor $X \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$, we define the following quantity, which we call the* compatibility norm *of X:*

$$\|X\|_c := \inf \left\{ \Big\| \sum_{j=1}^{K} H_j \Big\|_\infty \, : \, X = \sum_{j=1}^{K} z_j \otimes H_j, \;\; s.t. \;\; \forall j \in [K], \, \|z_j\|_\infty \leq 1 \text{ and } H_j \geq 0 \right\}. \tag{9.4}$$

Note that in the case of a single matrix ($N = 1$) we have $\|(X_1)\|_c = \|X_1\|_\infty$, and that, in general, we have

$$\|X\|_c = \inf \left\{ t \, : \, X = \sum_{j=1}^{K} z_j \otimes H_j, \;\; \text{s.t.} \;\; \sum_{j=1}^{K} H_j = tI_d \text{ and } \forall j \in [K], \, \|z_j\|_\infty = 1, H_j \geq 0 \right\}.$$

Indeed, the condition $\|z_j\| = 1$ can be imposed by replacing a non-zero term $z_j \otimes H_j$ by $z_j/\|z_j\|_\infty \otimes \|z_j\|_\infty H_j$, while the condition $\sum_j H_j = tI_d$ can be imposed by adding the term $0 \otimes (tI_d - \sum_j H_j)$ to the decomposition.

Proposition 9.7.2. *The $\|\cdot\|_c$ quantity is a tensor norm on $(\mathbb{R}^N, \|\cdot\|_\infty) \otimes (\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$.*

Proof. Let us start with the triangle inequality, $\|A + B\|_c \leq \|A\|_c + \|B\|_c$. Consider optimal decompositions

$$A = \sum_j z_j \otimes H_j$$
$$B = \sum_k w_k \otimes T_k$$

such that

$$\|A\|_c = \Big\| \sum_j H_j \Big\| \qquad \text{and} \qquad \|B\|_c = \Big\| \sum_k T_k \Big\|.$$

Then,

$$A + B = \sum_j z_j \otimes H_j + \sum_k w_k \otimes T_k$$

is a valid decomposition for $A + B$, hence

$$\|A + B\|_c \leq \Big\| \sum_j H_j + \sum_k T_k \Big\| \leq \Big\| \sum_j H_j \Big\| + \Big\| \sum_k T_k \Big\| = \|A\|_c + \|B\|_c.$$

The scaling equality $\|\lambda A\|_c = |\lambda|\|A\|_c$ is straightforward, and left to the reader. Consider now A such that $\|A\|_c = 0$. Then, for all $\varepsilon > 0$, there is a finite decomposition $A = \sum_j z_j \otimes H_j$ such that $\| \sum_j H_j\| \leq \varepsilon$. We have then, for all $x \in [N]$,

$$\|A_x\| = \Big\| \sum_j z_j(x) H_j \Big\| \leq \Big\| \sum_j |z_j(k)| H_j \Big\| \leq \Big\| \sum_j H_j \Big\| \leq \varepsilon.$$

Taking $\varepsilon \to 0$ shows that $A_x = 0$ for all x, and thus $A = 0$.

The fact that the compatibility norm is bounded by the injective and projective norms is established in [87, Proposition 3.3]. Finally, let us show that $\| \cdot \|_c$ factorizes on simple tensors. To this end, consider a (non-zero) product tensor $A = w \otimes T$ with $\|w\|_\infty = 1$ (this can always be enforced by absorbing the norm of w into T). On the one hand, we have

$$\|A\|_c \leq \|T\| = \|w\|_\infty \|T\|,$$

establishing one inequality. Consider now an optimal decomposition

$$w \otimes T = \sum_j z_j \otimes H_j$$

with $\|z_j\|_\infty \leq 1$, $H_j \geq 0$, and $\|w \otimes T\|_c = \| \sum_j H_j\|$. Consider an index $k \in [N]$ such that $\|w\|_\infty = |w(k)|$. We have then $w(k)T = \sum_j z_j(k) H_j$ and thus

$$\|w\|_\infty \|T\| = \Big\| \sum_j z_j(k) H_j \Big\| \leq \Big\| \sum_j |z_j(k)| H_j \Big\| \leq \Big\| \sum_j H_j \Big\| = \|w \otimes T\|_c,$$

finishing the proof. □

We specialize now [86, Theorem 9.2] to the case of quantum mechanics, showing that the compatibility norm from Definition 9.7.1.

Theorem 9.7.3. *Let $A = (A_1, \ldots, A_N)$ be a N-tuple of self-adjoint $d \times d$ complex matrices. Then:*

1. *A is a collection of dichotomic quantum observables (i.e. $\|A_i\|_\infty \leq 1$ $\forall i$) if and only if $\|A\|_\varepsilon \leq 1$, where $\| \cdot \|_\varepsilon$ quantity is the $\ell_\infty^N \otimes_\varepsilon S_\infty^d$ tensor norm.*
2. *A is a collection of* compatible *dichotomic quantum observables if and only if $\|A\|_c \leq 1$.*

Proof. The first statement is a direct consequence of (9.3). For the second statement, we shall prove the two implications separately.

First, consider compatible dichotomic observables $A_1, \ldots, A_N$, and their joint POVM X, having $X_\varepsilon \geq 0$ indexed by sign vectors $\varepsilon \in \{\pm 1\}^N$, such that

$$\forall i \in [N], \forall s \in \{\pm 1\}, \qquad E_i^s = \frac{I_d + sA_i}{2} = \sum_{\varepsilon \in \{\pm 1\}^N : \varepsilon_i = s} X_\varepsilon.$$

In particular, we have, for all $i \in [N]$,

$$A_i = -I_d + 2 \sum_{\varepsilon \in \{\pm 1\}^N : \varepsilon_i = +1} X_\varepsilon$$

and thus

$$A = \sum_{i=1}^N e_i \otimes A_i = \sum_{i=1}^N (-e_i) \otimes I_d + 2 \sum_{\varepsilon \in \{\pm 1\}^N} \left(\sum_{i\,:\,\varepsilon_i = +1} e_i \right) \otimes X_\varepsilon$$

$$= \sum_{\varepsilon \in \{\pm 1\}^N} \left(2 \sum_{i\,:\,\varepsilon_i = +1} e_i - \sum_i e_i \right) \otimes X_\varepsilon$$

$$= \sum_{\varepsilon \in \{\pm 1\}^N} \underbrace{\left(\sum_i \varepsilon_i e_i \right)}_{=: z_\varepsilon} \otimes X_\varepsilon.$$

We have thus obtained above a decomposition of the tensor A with 2^N terms, $\|z_\varepsilon\|_\infty = 1$ and $\sum_\varepsilon X_\varepsilon = I_d$, proving that $\|A\|_c \leq 1$.

For the reverse implication, start with a decomposition $A = \sum_j z_j \otimes H_j$ with $\|z_j\|_\infty \leq 1$, $H_j \geq 0$ and $\sum_j H_j = I_d$. One can recover the observables and the from this decomposition:

$$A_i = \sum_j z_j(i) H_j \quad \text{and} \quad E_i^\pm = \sum_j \frac{1 \pm z_j(i)}{2} H_j.$$

One recognizes in the expression above the description of the compatibility of the POVMs $(E_i^+, E_i^-)_{i \in [N]}$ as post-processing from Proposition 9.3.3:

$$E_i^\pm = \sum_j p_i(\pm|j) H_j,$$

where the conditional probabilities p_i are given by

$$p_i(\pm|j) = \frac{1 \pm z_j(i)}{2} \in [0, 1].$$

□

The compatibility norm of a tensor A is related to the noise parameter Γ from Definition 9.3.7. The following proposition provides an *operational interpretation* of the compatibility norm $\|A\|_c$, as the inverse of the minimal quantity of white noise that needs to be mixed in the measurements A in order to render them compatible.

Proposition 9.7.4. *For any N-tuple of observables $A = (A_1, A_2, \ldots, A_N) \neq 0$,*

$$\Gamma(A) = \frac{1}{\|A\|_c}.$$

Proof. Note first that, on the level of observables, adding noise to a dichotomic measurement corresponds to scaling:

$$A^\eta = \left[\eta E + (1-\eta) \frac{I}{2} \right] - \left[I - \eta E - (1-\eta) \frac{I}{2} \right] = 2\eta E - \eta I = \eta A.$$

Hence,

$$\begin{aligned} \Gamma(A) &= \max\{\eta \,:\, (A_1^\eta, \ldots, A_N^\eta) \text{ compatible}\} \\ &= \max\{\eta \,:\, \|A_1^\eta, \ldots, A_N^\eta\|_c \leq 1\} \\ &= \max\{\eta \,:\, \eta \|A_1, \ldots, A_N\|_c \leq 1\} \\ &= \frac{1}{\|A\|_c}. \end{aligned}$$

□

Example 9.7.5. *Let us consider the example of the unbiased Pauli measurements,*

$$\frac{1}{2}(I_2 \pm x\sigma_X), \quad \frac{1}{2}(I_2 \pm y\sigma_Y), \quad \frac{1}{2}(I_2 \pm z\sigma_Z),$$

where $(x, y, z) \in [0, 1]^3$ *are real parameters describing the noise in the measurements. These three POVMs correspond to the observables*

$$A_X = x\sigma_X, \quad A_Y = y\sigma_Y, \quad A_Z = z\sigma_Z.$$

It is known [135, 136] that these observables are compatible if and only if $x^2 + y^2 + z^2 \leq 1$, *hence*

$$\|(A_X, A_Y, A_Z)\|_c = \sqrt{x^2 + y^2 + z^2} = \|(x, y, z)\|_2.$$

9.8 The relation between nonlocality and incompatibility

Having introduced the main conceptual definitions of the *(in)compatibility norm* and *M-Bell-locality norm* in the previous sections that formalize the compatibility and the nonlocality physical notions for fixed measurements on Alice's side apparatus and invertible non-local games M, we bring together and compare the two norms. In this section we introduce *the main theorems* of the paper. It was shown in [9] that the two notions are equivalent in the case of the CHSH game. Using the framework of tensor norms, we shall give a quantitative and precise answer to the following question:

When is measurement incompatibility equivalent to nonlocality for general games?

It turns out that the answer to this question is given by a comparison between the *compatibility norm* and the *M-Bell-locality norm*. For the reader's convenience, we recall the definitions of the two tensor norms that we introduced in the Sections 9.6 and 9.7, in relation to, respectively, Bell inequality violations and measurement incompatibility.

- The *M-Bell-locality norm* (see Definition 9.6.1 and Theorem 9.6.10)

$$\|A\|_M := \sup_{\|\psi\|=1} \sup_{\|B_y\|\leq 1} \left\langle \psi \middle| \sum_{x,y=1}^{N} M_{xy}\, A_x \otimes B_y \middle| \psi \right\rangle = \lambda_{\max} \left[\sum_{y=1}^{N} \left| \sum_{x=1}^{N} M_{xy}\, A_x \right| \right].$$

- The *compatbility norm* (see Definition 9.7.1 and Theorem 9.7.3)

$$\|A\|_c := \inf \left\{ \Big\| \sum_{j=1}^{K} H_j \Big\|_\infty \; : \; A = \sum_{j=1}^{K} z_j \otimes H_j, \;\; \text{s.t. } \; \forall j \in [K], \, \|z_j\|_\infty \leq 1 \text{ and } H_j \geq 0 \right\}.$$

In what follows, we shall compare these two norms, in order to relate, in a quantitative manner, the two fundamental physical phenomena of Bell nonlocality and measurement incompatibility.

We start with a reformulation, using the language of tensor norms, of the following well established fact: an observed *violation of the Bell inequality M* implies necessarily the *incompatibility* of Alice's measurements. Mathematically, this corresponds to upper bounding the M-Bell-locality norm of Alice's measurements by their compatibility norm.

Theorem 9.8.1. *Consider a N-input, 2-output non-local ineurtible game M, corresponding to a matrix $M \in \mathcal{M}_N(\mathbb{R})$. Then, for any N-tuple of self-adjoint matrices $A = (A_1, \ldots, A_N)$, we have*

$$\|A\|_M \leq \|A\|_c \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} = \|A\|_c\, \beta(M). \tag{9.5}$$

In particular, if Alice's measurements A are M-Bell-non-local (in the sense of Definition 9.6.3), then they must be incompatible.

Proof. Let us consider the optimal decomposition $\|A\|_c = \|\sum_{j=1}^N C_j\|_\infty$ with $A = \sum_{j=1}^N z_j \otimes C_j$, $\|z_j\|_\infty \leq 1$ and $C_j \geq 0$ for all j. Thus we have $A_x = \sum_{j=1}^N z_j(x) C_j$.
We compute the upper bound of the M-Bell-locality norm $\|A\|_M$.

$$\|A\|_M = \lambda_{\max}\left[\sum_{y=1}^N \left|\sum_{x=1}^N M_{xy}\, A_x\right|\right] = \lambda_{\max}\left[\sum_{y=1}^N \left|\sum_{j=1}^N \sum_{x=1}^N M_{xy}\, z_j(x)\, C_j\right|\right]$$
$$\leq \lambda_{\max}\left[\sum_{j=1}^N \sum_{y=1}^N \left|\sum_{x=1}^N M_{xy}\, z_j(x)\right| C_j\right] = \lambda_{\max}\left[\sum_{j=1}^N \sum_{y=1}^N \sum_{x=1}^N \varepsilon_y\, M_{xy}\, z_j(x)\, C_j\right]$$

We have used

$$\sum_{y=1}^N \left|\sum_{x=1}^N M_{xy}\, z_j(x)\right| = \sum_{y=1}^N \sum_{x=1}^N \varepsilon_y\, M_{xy}\, z_j(x)$$

with $\varepsilon = \{\pm 1\}^N$.

Then we have

$$\|A\|_M \leq \lambda_{\max}\left[\sum_{j=1}^N \sum_{y=1}^N \sum_{x=1}^N \varepsilon_y\, M_{xy}\, z_j(x)\, C_j\right] \leq \lambda_{\max}\left[\sum_{j=1}^N C_j\, \|M\|_{\ell_1 \otimes_\varepsilon \ell_1}\right]$$
$$= \lambda_{\max}\left[\sum_{j=1}^N C_j\right] \|M\|_{\ell_1 \otimes_\varepsilon \ell_1} = \|A\|_c \|M\|_{\ell_1 \otimes_\varepsilon \ell_1}$$

where $\|M\|_{\ell_1 \otimes_\varepsilon \ell_1} = \sup_{\|\varepsilon\|_\infty \leq 1, \|z_j\|_\infty \leq 1} \langle M, z_j \otimes \varepsilon \rangle$. □

In the following we will show, for *invertible Bell functionals*, that the compatibility is upper bounded by the M-Bell-locality norm.

Theorem 9.8.2. *Consider a N-input, 2-output non-local game M, corresponding to an* invertible *matrix $M \in \mathcal{M}_N(\mathbb{R})$. Then, for any N-tuple of self-adjoint matrices $A = (A_1, \ldots, A_N)$, we have*

$$\|A\|_c \leq \|A\|_M \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}. \tag{9.6}$$

Proof. Let us consider

$$C_y = \sum_{x=1}^N M_{xy} A_x = (M^\top A)_y \implies A_x = ((M^\top)^{-1} C)_x = \sum_{y=1}^N (M^{-1})_{y,x} C_y.$$

Let us also consider the following decomposition of $A = \sum_{x=1}^N e_x \otimes A_x$ with e_x the canonical basis vectors. We have

$$A = \sum_{x=1}^N e_x \otimes A_x = \sum_{y=1}^N \sum_{x=1}^N (M^{-1})_{y,x} e_x \otimes C_y$$
$$= \sum_{y=1}^N \left[\sum_{x=1}^N (M^{-1})_{y,x} e_x\right] \otimes C_y^+ + \sum_{y=1}^N \left[-\sum_{x=1}^N (M^{-1})_{y,x} e_x\right] \otimes C_y^-$$
$$= \sum_{y=1}^N e'_y \otimes C_y^+ + \sum_{y=1}^N -e'_y \otimes C_y^-$$

where we have decomposed $C_y = C_y^+ - C_y^-$ into positive and negative parts $C_y^\pm \geq 0$ for all $y \in [N]$ and $e'_y := \sum_{x=1}^N (M^{-1})_{y,x} e_x$.

Observe that

$$\|e'_y\|_\infty = \Big\| \sum_{x=1}^N (M^{-1})_{y,x} e_x \Big\|_\infty = \Big\| (M^{-1}e)_y \Big\|_\infty \leq \|M^{-1}\|_\infty \|e\|_\infty = \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}$$

where we recall by the Definition 10.2.12 that

$$\|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} := \sup_{a,b \in \mathbb{B}(\ell_1^N(\mathbb{R}))} |\langle M^{-1}, a \otimes b \rangle| = \max_{i,j} |(M^{-1})_{i,j}| = \|M^{-1}\|_\infty.$$

With the norm formulation above, $\|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}$ and $\|M^{-1}\|_\infty$ are equal if we consider M^{-1} in its matrix or tensor representations. Hence, we have

$$\|e'_y\|_\infty \leq \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}.$$

We consider now the normalised vectors

$$a_y := \frac{e_y}{\|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}} \in \mathbb{B}(\ell_\infty(\mathbb{R}^N))$$

By normalising the vectors one has

$$A = \sum_{y=1}^N \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}\, a_y \otimes C_y^+ - \sum_{y=1}^N \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}\, a_y \otimes C_y^-$$

We recognize above a valid decomposition of the tensor A as in Eq. (9.4). Hence

$$\begin{aligned} \|A\|_c &\leq \Big\| \sum_{y=1}^N \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} (C_y^+ + C_y^-) \Big\|_\infty \\ &= \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \lambda_{max}\Big(\sum_{y=1}^N \Big| C_y \Big| \Big) = \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|A\|_M. \end{aligned}$$

□

Putting together Theorems 9.8.1 and 9.8.2, we recover the main result from [9]: for $N = 2$ and the CHSH matrix

$$M_{\text{CHSH}} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

we have

$$\beta(M_{\text{CHSH}}) = 1 \qquad \text{and} \qquad (M_{\text{CHSH}})^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

It follows thus, from Eqs. (10.5.9) and (10.2) that

$$\|\cdot\|_c = \|\cdot\|_{M_{\text{CHSH}}}. \tag{9.7}$$

Remark 9.8.3. *We have seen in Section 9.5 that for the CHSH game we have*

$$\|A\|_{M_{CHSH}} = \frac{1}{\Gamma(A)}.$$

One can see also that within Proposition 9.7.4 we have, with respect to the compatibility norm,

$$\|A\|_c = \frac{1}{\Gamma(A)}.$$

In what follows, we compare the compatibility norm and the Bell-locality norm in two special settings, a modified CHSH game, and the pure correlation I_{3322} game. As we shall see, in these situations, the two norms are different.

Let us first consider the following modified CHSH game defined by the following matrix

$$M_t := \begin{bmatrix} 1 & 1 \\ 1 & -t \end{bmatrix},$$

where t is a real parameter taking values in $\mathbb{R} \setminus \{-1\}$, such that the matrix M_t is invertible. We start by normalizing the matrix M_t such that its classical bias β is equal to 1. A simple calculation shows that

$$\beta(M_t) := \begin{cases} 3-t & \text{for} \quad t \leq 1 \\ 1+t & \text{for} \quad t > 1 \end{cases} = 2 + |t-1|.$$

We are thus going to work with the normalized version

$$M'_t = \frac{1}{\beta(M_t)} \begin{bmatrix} 1 & 1 \\ 1 & -t \end{bmatrix},$$

for which $\beta(M'_t) = 1$. We consider the following pair of spin observables

$$A := (\sigma_X, y\sigma_Y),$$

where $y \in [-1, 1]$ is a parameter we shall vary. These two observables correspond to, respectively, a sharp measurement in the eigenbasis of σ_X and a noisy measurement in the eigenbasis of σ_Y.

In the following, we calculate $\|A\|_c$ and $\|A\|_{M'_t}$, for different values of the parameters t and y. Since the $t = 1$ value corresponds to the $CHSH$ game (for which $\|\cdot\|_c = \|\cdot\|_{M_{\text{CHSH}}}$), the compatibility norm reads

$$\|A\|_c = \|A\|_{M'_{t=1}} = \|A\|_{M_{\text{CHSH}}} = \lambda_{\max}\left[\sum_{y=1}^{2}\left|\sum_{x=1}^{2} M_{xy}A_x\right|\right] = \frac{1}{2}\lambda_{\max}\big[|\sigma_X + y\sigma_Y| + |\sigma_X - y\sigma_Y|\big].$$

A simple calculation shows that

$$\|A\|_c = \sqrt{1+y^2} =: r.$$

We now compute $\|A\|_{M'_t}$ for the normalized modified CHSH game:

$$\|A\|_{M'_t} = \frac{1}{2+|t-1|}\lambda_{\max}\big[|\sigma_X + y\sigma_Y| + |\sigma_X - ty\sigma_Y|\big] = \frac{r_t + r}{2+|t-1|}.$$

with $r_t := \sqrt{1+(yt)^2}$; above, we have used the following fact:

$$\forall x, y \in \mathbb{R}, \qquad |x\sigma_X + y\sigma_Y| = \left|\begin{bmatrix} 0 & x - \mathrm{i}y \\ x + \mathrm{i}y & 0 \end{bmatrix}\right| = \sqrt{x^2+y^2}\, I_2.$$

. We plot the norm $\|A\|_{M'_t}$ in Figure 9.4, the region t and y where Alice observes a Bell inequality violation $\|A\|_{M'_t} > \beta(M_t) = 1$ in Figure 9.5, and the ratio of the two norms in Figure 9.6. Note that the plot for $t = 1$ corresponds to the CHSH game: the two norms are equal (see Eq. (9.7)). At $y = 1$, Alice's measurements are sharp: $A = (\sigma_X, \sigma_Y)$. One observes violations of the game M'_t for the parameter values

$$\|A_{y=1}\|_{M'_t} > 1 \iff t > \frac{9-4\sqrt{2}}{7} =: t_*.$$

The values at $y = 0$ also have a special meaning, since, in this case,

$$A = (\sigma_X, 0) = (1, 0) \otimes \sigma_X.$$

By the tensor norm property of the compatibility norm (see Proposition 9.7.2), we have

$$\|A\|_c = \|(1, 0)\|_{\ell_\infty} \cdot \|\sigma_X\|_{S_\infty} = 1.$$

Similarly, the tensor norm property of the Bell-locality norm yields

$$\|A\|_{M'_t} = \|(1, 0)\|_{M'_t} \cdot \|\sigma_X\|_{S_\infty} = \frac{2}{\beta(M_t)} \cdot 1 = \frac{2}{2 + |t - 1|} \leq 1.$$

In Figure 9.6, the dashed curve corresponds to the limit $|t| \to \infty$, in which case

$$\lim_{|t|\to\infty} \frac{\|A\|_{M'_t}}{\|A\|_c} = \frac{|y|}{\sqrt{1 + y^2}}.$$

Finally, the dotted line corresponds to the game M'_t for $t = -1$. This game is not invertible, so the quantity $\| \cdot \|_{M'_{-1}}$ is not a norm.

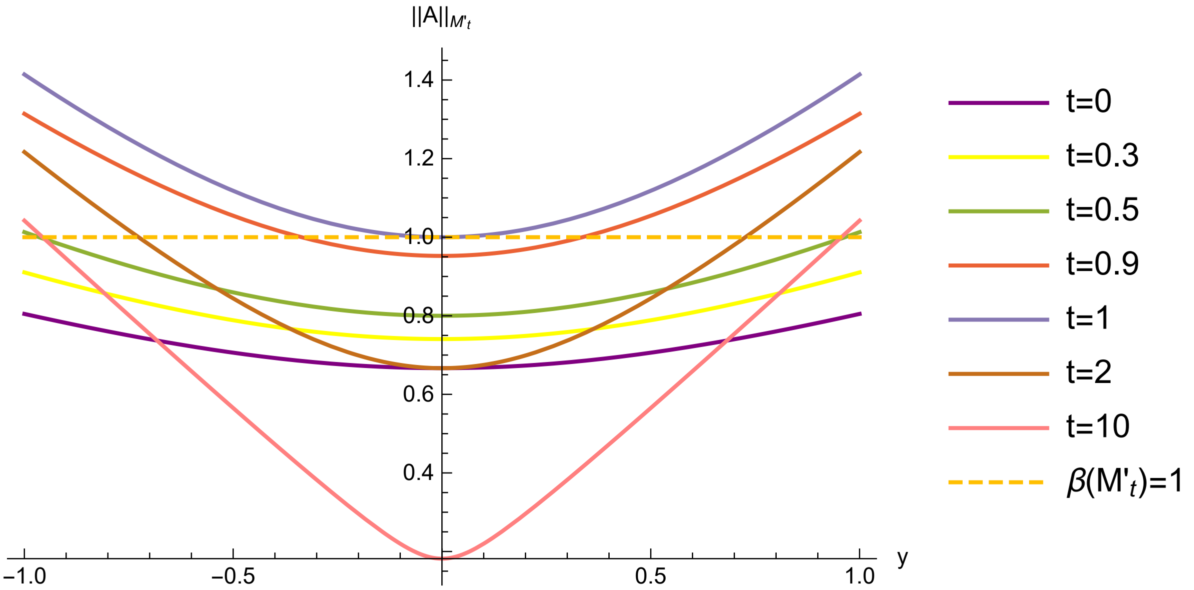


Figure 9.4: The norm $\|A\|_{M'_t}$ for $y \in [-1, 1]$ and different value of t. The measurements A are σ_X and $y\sigma_Y$, a noisy version of σ_Y. For $t = 1$ (the CHSH game), one observes violations (i.e. $\|A\|_{M'_t} > \beta(M'_t) = 1$) for every value of $y \in [-1, 1]$.

In the same spirit as the example above we shall now analyze another deformation of the CHSH game that was considered in [137]. In the following we shall recall the game, and analyse it as the game considered before with the tools that we introduced.

The deformation of the CHSH game that was considered in [137], was given by

$$G(p, q) = \begin{bmatrix} p\,q & p\,(1 - q) \\ q\,(1 - p) & -(1 - q)\,(1 - p) \end{bmatrix},$$

where $p, q \in [0, 1]^2$. Note that this matrix is invertible for all $(p, q) \in (0, 1)^2$.

In the following we give the classical bias $\beta(G(p, q))$ for different $p, q \in [0, 1]^2$.

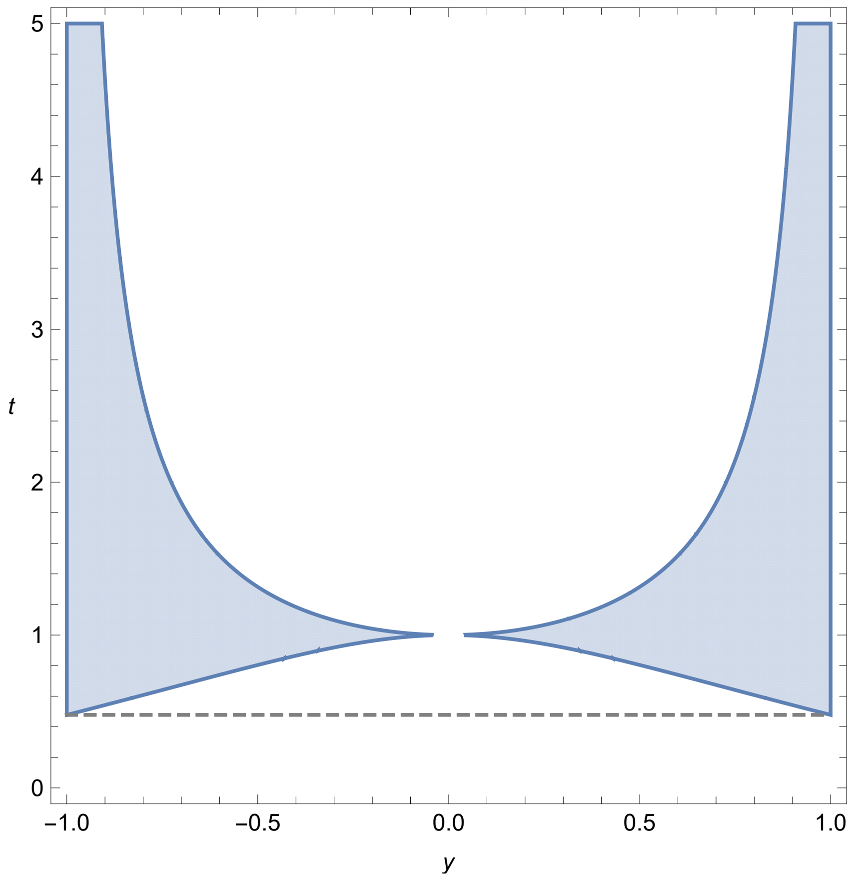


Figure 9.5: The filled region corresponds to the set of parameters (y, t) for which Alice's measurements (σ_X and $y\sigma_Y$) are Bell non-local (for the game M'_t): $\|A\|_{M'_t} > 1 = \beta(M'_t)$. Note that for $t > t_* = (9 - 4\sqrt{2})/7$, the game M'_t does not allow quantum violations when Alice's measurements are of the form $A = (\sigma_X, y\sigma_Y)$.

- For p and q satisfying $p, q \geq \frac{1}{2}$, the classical bias of the game $\beta(G(p,q))$ is given by

$$\beta(G(p,q)) = 1 - 2\,(1-p)\cdot(1-q)$$

- For p and q satisfying $p \leq \frac{1}{2}, q \geq \frac{1}{2}$, the classical bias of the game $\beta(G(p,q))$ is given by

$$\beta(G(p,q)) = 1 - 2\,p\cdot(1-q)$$

- For p and q satisfying $q \leq \frac{1}{2}, p \geq \frac{1}{2}$, the classical bias of the game $\beta(G(p,q))$ is given by

$$\beta(G(p,q)) = 1 - 2\,q\cdot(1-p)$$

- For p and q satisfying $p, q \leq \frac{1}{2}$, the classical bias of the game $\beta(G(p,q))$ is given by

$$\beta(G(p,q)) = 1 - 2p\cdot q$$

.

Remark 9.8.4. *The classical bias of the game $\beta(G(p,q))$ for $p, q \geq \frac{1}{2}$ was already shown in [137].*

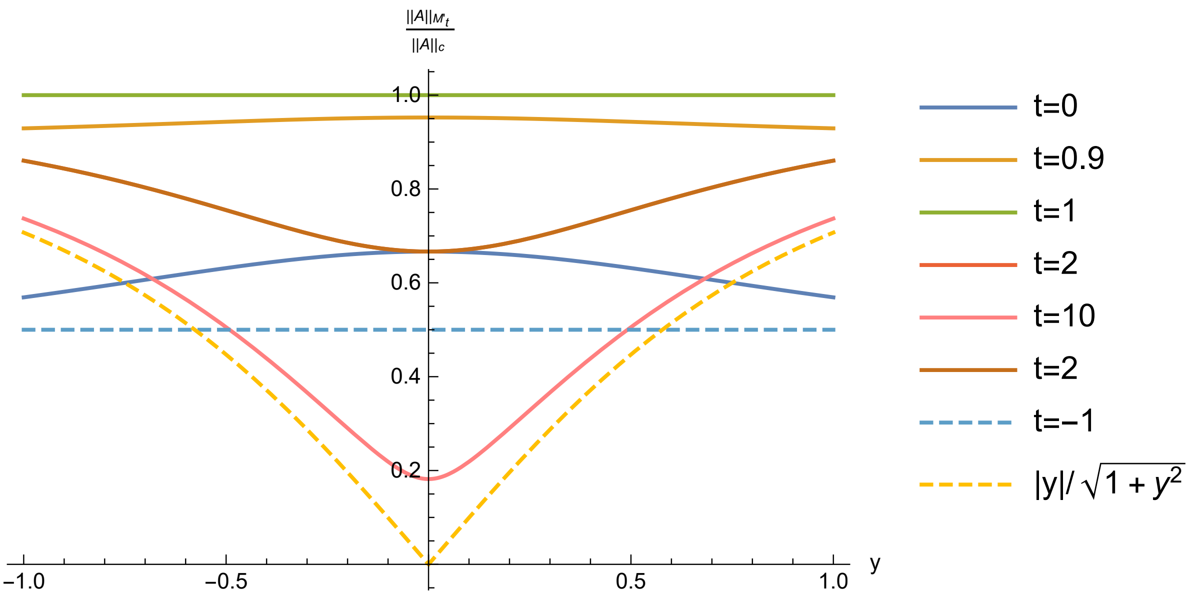


Figure 9.6: The ratio $\|A\|_{M'_t} / \|A\|_c$ for $y \in [-1, 1]$ and different value of t. We notice that the ration is always smaller than 1, except for $t = 1$, which is the CHSH game.

To give the classical bias of the game $\beta(G(p, q))$ for different $p, q \in [0, 1]^2$, one shall use $\min(\cdot, \cdot)$ defined as

$$\min(x, y) := \frac{1}{2}(x + y - |x - y|).$$

with $x, y \in \mathbb{R}$.

One can easily check that for $p \in [0, 1]$ we have

$$\min(p, 1-p) = \frac{1}{2}(1 - |2p - 1|) = \begin{cases} 1-p & \text{for} \quad p \geq \frac{1}{2} \\ p & \text{for} \quad p \leq \frac{1}{2} \end{cases}$$

The same results hold for $\min(q, 1-q)$ with $q \in [0, 1]$.

It can be easily seen that the classical bias of the game for $p, q \in [0, 1]^2$ is given by:

$$\beta(G(p, q)) = 1 - 2 \cdot \min(p, 1-p) \min(q, 1-q).$$

In our setting, we shall consider the normalised game $G'(p, q)$ for all $p, q \in [0, 1]^2$

$$G'(p, q) = \frac{1}{\beta(G(p, q))} \begin{bmatrix} p\,q & p\,(1-q) \\ q\,(1-p) & -(1-q)\,(1-p) \end{bmatrix}.$$

As in the example above we consider the following pair of spin observables

$$A := (\sigma_X, y\sigma_Y),$$

where $y \in [-1, 1]$ is a parameter we shall vary. In the following we will compute the $\|A\|_{G'(p,q)}$.

$$\begin{aligned} \|A\|_{G'_{p,q}} &:= \lambda_{\max}\left[\sum_{y=1}^{2}\left|\sum_{x=1}^{2} G'(p, q)_{x,y} A_x\right|\right] \\ &= \frac{1}{|\beta(G(p, q))|}\lambda_{\max}\Big(\Big|p\,q\,\sigma_X + y\,q\,(1-p)\sigma_Y\Big| + \Big|p\,(1-q)\sigma_X - y\,(1-p)\,(1-q)\,\sigma_Y\Big|\Big). \end{aligned}$$

A simple calculation shows that

$$\|A\|_{G'(p,q)} = \frac{1}{|1-2\min(p,1-p)\min(q,1-q)|}\left(\left[p^2\,q^2+y^2\,q^2\,(1-p)^2\right]^{\frac{1}{2}}+\left[p^2\,(1-q)^2+y^2\,(1-p)^2\,(1-q)^2\right]^{\frac{1}{2}}\right).$$

Note that for $p = 1/2$, we have a simplification:

$$\|A\|_{G'(1/2,q)} = \frac{\sqrt{1+y^2}}{2\max(q,1-q)} = \frac{\|A\|_c}{2\max(q,1-q)}.$$

We plot in Figure 9.7 the set of pairs (p,q) such that $\|A\|_{G'(p,q)} > 1$, that is the game parameter region where Alice observes a Bell violation for different values of $y \in [0,1]$. In Figure 9.8 we plot the norm $\|A\|_{G'(p,q)}$ while in Figure 9.9 the ratio of $\|A\|_{G'(p,q)}/\|A\|_c$ for fixed value of p and q .

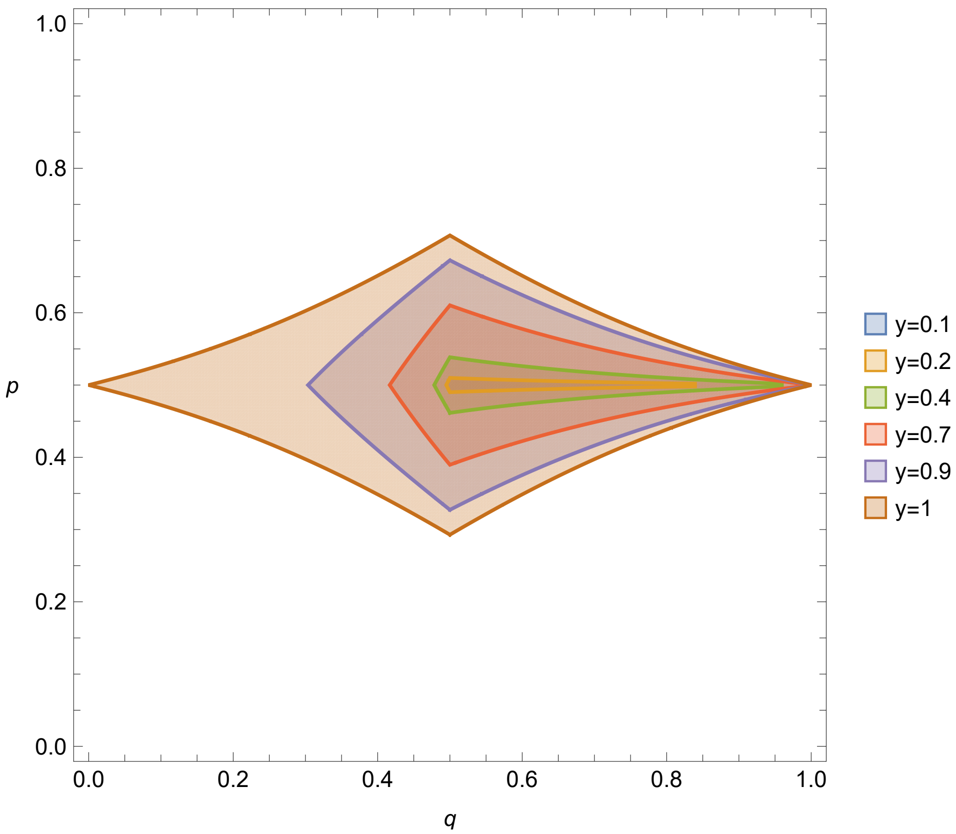


Figure 9.7: The p,q region where Alice observe a violation $\|A\|_{G'(p,q)} > 1$ for different value of y.

We now move on to the last example, the pure correlation part of the I_{3322} tight Bell inequality (here, $N = 3$):

$$M_{3322} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

The inverse of the matrix above has entries with absolute value 2, so our main result does not apply. Indeed, one can see that

$$\|s(\sigma_X,\sigma_Y,\sigma_Z)\|_c \leq 1 \iff s \leq \frac{1}{\sqrt{3}},$$

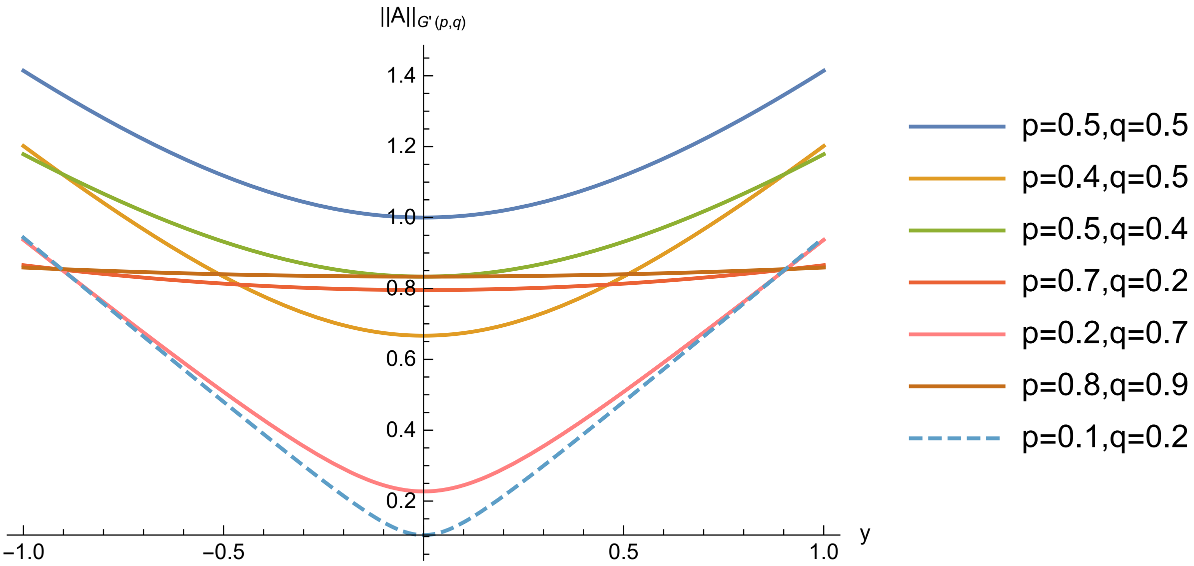


Figure 9.8: The norm $\|A\|_{G'(p,q)}$ for $y \in [-1,1]$ and different value of p and q.

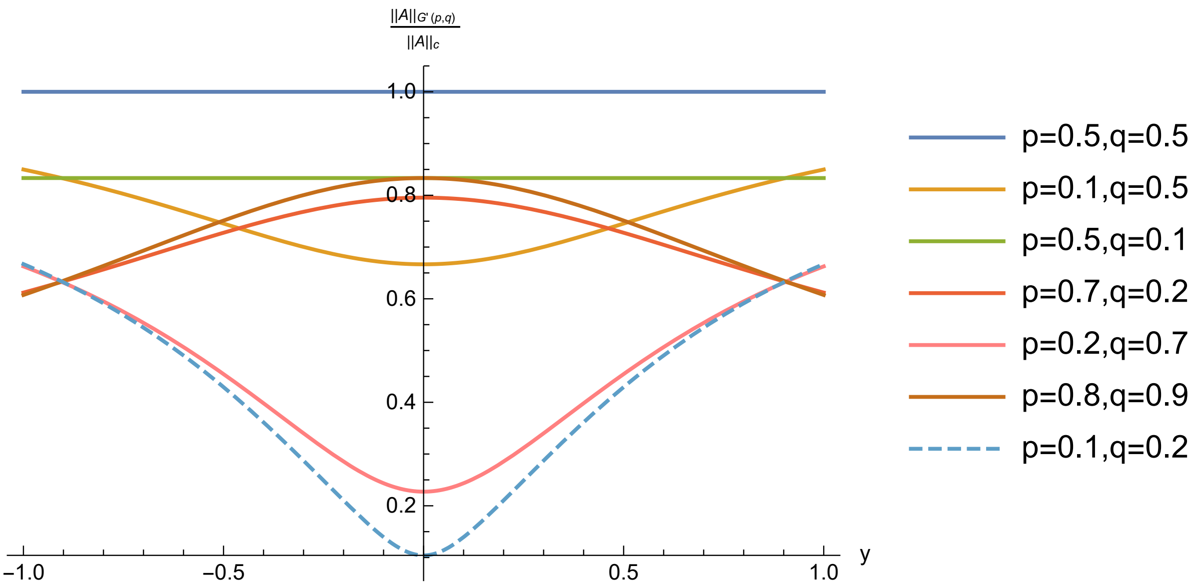


Figure 9.9: The ratio $\|A\|_{G'(p,q)}/\|A\|_c$ for $y \in [-1,1]$ and different value of p and q.

while

$$\|s(\sigma_X,\sigma_Y,\sigma_Z)\|_{M_{3322}} \le 1 \iff s \le \frac{4}{\sqrt{2}+2\sqrt{3}} > \frac{1}{\sqrt{3}}.$$

This shows that, for tensors in the positive direction $(\sigma_X,\sigma_Y,\sigma_Z)$, for parameter values

$$s \in \left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{2}+2\sqrt{3}}\right],$$

we have

$$\|s(\sigma_X,\sigma_Y,\sigma_Z)\|_{M_{3322}} \le 1 < \|s(\sigma_X,\sigma_Y,\sigma_Z)\|_c,$$

so there exist incompatible dichotomic Pauli measurements which do not violate the pure correlation I_{3322} Bell inequality [89].

9.9 Non-local games which characterize incompatibility

Up to this point, we have seen the following two inequalities relating the M-Bell-locality norm $\|\cdot\|_M$ and the compatibility norm $\|\cdot\|_c$ of a tuple of dichotomic quantum measurements:

$$\|A\|_M \leq \|A\|_c \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \qquad \text{and} \qquad \|A\|_c \leq \|A\|_M \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}.$$

In this section, we ask for which (invertible) non-local games M, these two inequalities, used together, allow us to conclude that $\|\cdot\|_M = \|\cdot\|_c$. Such an equality would prove a strong equivalence of Bell inequality violations and incompatibility for the game M, in the spirit of [9].

First, note that, for an invertible game M and a non-zero tuple of measurements A, we have

$$\|A\|_M \leq \|A\|_c \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \leq \|A\|_M \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N},$$

hence

$$\|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \geq 1. \tag{9.8}$$

In order to deduce that $\|\cdot\|_M = \|\cdot\|_c$, one requires

$$\beta(M) = \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} = 1 \qquad \text{and} \qquad \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} = 1.$$

Up to rescaling, this is equivalent to requiring that the inequality (10.3) should be saturated. We now study the equality case in (10.3), which can be seen as an “uncertainty relation” for the non-local game M.

Let us first show that (10.3) cannot be saturated for $N \geq 3$. We recall and use a definition from [138] to understand the ratio of $\|\cdot\|_{X\otimes_\pi Y}$ and $\|\cdot\|_{X\otimes_\varepsilon Y}$ for two given Banach spaces X and Y.

Definition 9.9.1. *[138] Given two finite-dimensional Banach spaces X and Y. There will always exist a constant $1 \leq C < \infty$ such that:*

$$\|\cdot\|_{X\otimes_\varepsilon Y} \leq \|\cdot\|_{X\otimes_\pi Y} \leq C\|\cdot\|_{X\otimes_\varepsilon Y}$$

One denotes $\rho(X,Y)$ the smallest C satisfying this inequality. Equivalently one has

$$\rho(X,Y) = \sup_{0\neq z\in X\otimes Y} \frac{\|z\|_{X\otimes_\pi Y}}{\|z\|_{X\otimes_\varepsilon Y}}$$

We recall one of the important properties of $\rho(X,Y)$ in the case of ℓ_1 and ℓ_∞ spaces.

Proposition 9.9.2. *[138, Proposition 13] For all $N \geq 2$, we have*

$$\rho(\ell_1^N, \ell_1^N) = \rho(\ell_\infty^N, \ell_\infty^N) \leq \sqrt{2N}.$$

With the help of the definition of $\rho(X,Y)$ and proposition above, we can[1] improve the inequality (10.3).

Proposition 9.9.3. *Let M a real and invertible matrix. Then one has*

$$\|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \geq \frac{N}{\rho(\ell_\infty^N, \ell_\infty^N)} \geq \sqrt{\frac{N}{2}} \geq 1.$$

for $N \geq 2$. In particular, for $N \geq 3$, the last inequality above is strict.

[1] We thank Carlos Palazuelos for the proof of the proposition.

Proof. Let
$$N = \mathrm{Tr}[M^{-1}M] = \langle \tilde{M}, M\rangle_{H.S} \leq \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \|\tilde{M}\|_{\ell_\infty^N \otimes_\pi \ell_\infty^N}$$
Where $\tilde{M} := (M^{-1})^T$ Thus we have by definition 9.9.1 and we recall $\langle A, B\rangle_{H.S} := \mathrm{Tr}[A^* B]$
$$N \leq \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \|\tilde{M}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \rho(\ell_\infty^N, \ell_\infty^N)$$
Thus we have
$$1 \leq \frac{N}{\rho(\ell_\infty^N, \ell_\infty^N)} \leq \|M\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \|M^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}.$$
□

Having shown that inequality (10.3) cannot be saturated for $N \geq 3$, we now focus on the $N = 2$ case. We need the following lemma[2].

Lemma 9.9.4. *For any matrix $X \in \mathcal{M}_N(\mathbb{C})$ and for any unitary operators $U, V \in \mathcal{U}_N$, we have*
$$\|UXV^*\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|X\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \geq \frac{1}{N} |\det X|^{2/N}.$$
Equality holds if and only if both X and UXV^* are scalar multiples of Hadamard matrices.

Proof. Let $x \in \mathbb{C}^{N^2}$ be the vectorization of X; we have
$$\|X\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} = \max_{i,j \in [N]} |X_{ij}| = \|x\|_{\ell_\infty^{N^2}}.$$
Moreover, the vectorization of UXV^* is given by
$$y := (U \otimes \bar{V})x.$$
Using the unitarity of U, V and the fact that for all vectors $z \in \mathbb{C}^{N^2}$, $\|z\|_{\ell_\infty^{N^2}} \geq \|z\|_2 / N$, we have
$$\begin{aligned}\|UXV^*\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|X\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} &= \|y\|_{\ell_\infty^{N^2}} \|x\|_{\ell_\infty^{N^2}} \geq \frac{1}{N^2} \|y\|_2 \|x\|_2 \\ &= \frac{1}{N^2}\|x\|_2^2 = \frac{1}{N^2}\|X\|_2^2 = \frac{1}{N^2}\sum_{i=1}^N \sigma_i(X)^2.\end{aligned}$$
Above, $\sigma_i(X)$ denote the singular values of X. Using now the arithmetic mean-geometric mean (AM-GM) inequality, we have
$$\frac{1}{N}\sum_{i=1}^N \sigma_i(X)^2 \geq \left(\prod_{i=1}^N \sigma_i(X)^2\right)^{\frac{1}{N}} = |\det X|^{\frac{2}{N}}.$$
Hence
$$\|UXV^*\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|X\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \geq \frac{1}{N} |\det X|^{\frac{2}{N}},$$
proving the inequality.

In the derivation above, we have used three inequalities: the lower bound on the ℓ_∞ norm of the vectors x, y by their ℓ_2 norms, and the arithmetic and geometric inequality. If the former, equality holds iff the entries of, respectively, x and y are flat; this corresponds to the matrices X and UXV^* having, respectively, entries of identical absolute values. The latter corresponds to the singular values of X being identical, which corresponds to X being a scalar multiple of a unitary matrix. The announced equality condition follows from these considerations. □

[2] We thank Zbigniew Puchała for this result.

Recall that the Fourier matrix, also known as the discrete Fourier transform (DFT), is given by

$$F = N^{-1/2}\left[\omega^{ij}\right]_{i,j=0}^{N-1},$$

where $\omega = \exp(2\pi \mathrm{i}/N)$.

Proposition 9.9.5. *For $N = 2$, all the invertible Bell functionals $M \in \mathcal{M}_2(\mathbb{R})$ satisfying*

$$\|M^{-1}\|_{\ell^2_\infty \otimes_\varepsilon \ell^2_\infty} \|M\|_{\ell^2_1 \otimes_\varepsilon \ell^2_1} = 1$$

are of the form

$$M = a \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

with $a \in \mathbb{R}$, $a \neq 0$.

Proof. Since $M \in \mathcal{M}_2(\mathbb{R})$ one note that

$$\|M^{-1}\|_{\ell^2_\infty \otimes_\varepsilon \ell^2_\infty} = \frac{1}{|\det(M)|} \|M\|_{\ell^2_\infty \otimes_\varepsilon \ell^2_\infty}$$

and

$$\|M\|_{\ell^2_1 \otimes_\varepsilon \ell^2_1} = \|(T \otimes T)M\|_{\ell^2_\infty \otimes_\varepsilon \ell^2_\infty} = 2\,\|(F \otimes F)M\|_{\ell^2_\infty \otimes_\varepsilon \ell^2_\infty}$$

where $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \sqrt{2}F$, is an isometry $T : \ell^2_1 \to \ell^2_\infty$; geometrically, this corresponds to the fact that the unit ball of ℓ^2_1 (a diamond) is a scaled rotation of the unit ball of ℓ^2_∞ (a square). Now using the lemma above one has

$$\|M^{-1}\|_{\ell^2_\infty \otimes_\varepsilon \ell^2_\infty} \|M\|_{\ell^2_1 \otimes_\varepsilon \ell^2_1} = \frac{2}{|\det(M)|} \|M\|_{\ell^2_\infty \otimes_\varepsilon \ell^2_\infty} \|(F \otimes F)M\|_{\ell^2_\infty \otimes_\varepsilon \ell^2_\infty} \geq \frac{2}{|\det(M)|} \frac{|\det(M)|}{2} = 1$$

The equality holds as the lemma above iff M is a unitary and the entries of $(F \otimes F)M$ and M are flat (i.e. have the same absolute value). Then M is a multiple of a *Hadamard matrix*:

$$M = a \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad a \in \mathbb{R},\ a \neq 0.$$

□

Gathering Propositions 9.9.3 and 9.9.5, we obtain the following important characterization of non-local games M achieving equality in (10.3).

Theorem 9.9.6. *The only invertible non-local games $M \in \mathcal{M}_N(\mathbb{R})$ satisfying*

$$\|M^{-1}\|_{\ell^2_\infty \otimes_\varepsilon \ell^2_\infty} \|M\|_{\ell^2_1 \otimes_\varepsilon \ell^2_1} = 1$$

have two questions ($N = 2$) and are variants of the CHSH game: $M = aM_{CHSH}$ for some $a \neq 0$.

Note that saturating inequality (10.3) is just a *sufficient* condition for having $\|\cdot\|_M = \|\cdot\|_c$. We leave the general case open: for which non-local games M, does one have $\|\cdot\|_M = \|\cdot\|_c$?

9.10 Conclusion

Two of the most fundamental features of quantum mechanics are measurement incompatibility and the nonlocality of correlations. In this work, we address the relation between the two concepts within the natural framework of tensor norms. It was well-known that in order to observe correlation nonlocality in a Bell-type experiment, one should use incompatible measurements. Moreover, it was shown that in some particular cases, such as the CHSH game, incompatibility and the violation of the Bell inequality are equivalent. In the current paper, we introduced a natural framework in which one can directly compare the two notions. We have shown that the incompatibility is not in general equivalent to the nonlocality, by comparing two tensor norms.

Finally, let us address some questions we have left open. First and foremost, our setting is only adapted to dichotomic (2-outcome) POVMs; it would be interesting to extend the results in this paper to measurements with an arbitrary number of outcomes. The main obstacle here is encoding the outcomes of the g POVMs with more than 2 outcomes into a relevant tensor. In Section 9.9, we have shown that the two tensor norms, the one associated to a non-local game and the one associated to compatibility, cannot be shown to be equal using a simple chain of inequalities (except in the case of the CHSH game and its variants). The question whether the two norms can be equal (using other methods) is open. Here, one would need to rely on a more general argument instead of relying on some particular inequalities. Finally, our methods only cover XOR games with pure correlation terms; associating a tensor norm to more general games (such as the full I_{3322} game) is an interesting open problem.

Chapter 10

Resume en français[1]

Ce chapitre est un résumé de thèse en français où chacune des sections correspond à un chapitre du manuscrit. Nous reprendrons les deux concepts fondamentaux de la théorie quantique, à savoir la compatibilité des mesures quantiques, les violations de l'inégalité de Bell, ainsi le lien entre les deux. Il est bien connu que l'une des différences fondamentales entre la théorie quantique et la physique classique est l'existence de mesures incompatibles. Nous disons que deux mesures sont compatibles si nous pouvons les mesurer en même temps ; si elles ne le sont pas, nous disons qu'elles sont incompatibles. L'autre notion que nous aborderons dans cette thèse est la non-localité de la théorie quantique : au niveau quantique, le principe de localité n'est pas respecté. Dans la Section 10.1, nous donnerons une brève introduction à la théorie de l'information quantique. Dans la Section 10.2, nous donnerons une introduction à la théorie des normes tensorielles sur les espaces de Banach, celle-ci jouera un rôle crucial dans le cadre du lien entre la non-localité et l'incompatibilité. Dans la Section 10.3, nous donnons une introduction à la non-localité et aux inégalités de Bell, nous introduisons le cadre des jeux non-locaux. Dans la Section 10.4, nous introduirons la notion standard de compatibilité. Nous introduirons la nouvelle notion de dimension de compatibilité introduite dans [1], qui est utilisée pour analyser l'effet de la dimension de l'espace de Hilbert sur la compatibilité des mesures quantiques. Intuitivement, la dimension de compatibilité demande si, pour un tuple donné de mesures incompatibles dans un espace de Hilbert donné, on peut trouver un sous-espace de dimension réduite pour qu'elles deviennent compatibles. Nous présenterons différents types de modèles de bruit connus dans la littérature qui rendent compatibles des mesures incompatibles en construisant des mesures bruitées données comme une combinaison convexe de la mesure originale et d'un opérateur trivial. Nous explorerons également le lien entre le modèle de bruit et le clonage quantique approximatif. Comme application de la dimension de compatibilité, nous verrons que certaines mesures projectives bruitées incompatibles construites à partir de bases mutuellement non biaisées deviennent compatibles si on réduit la dimension de l'espace de Hilbert. Nous concluons cette section en introduisant la formulation de la compatibilité avec le formalisme de la norme tensorielle connue sous le nom de norme (tensorielle) de compatibilité . Dans la Section 10.5 nous allons faire le lien entre (in)compatibilité des mesures quantiques et non localité. Il a été montré dans [9] que l'incompatibilité est équivalente à la violation de l'inégalité de Bell pour le jeu CHSH, et la question de savoir si cette équivalence est valable pour d'autres jeux restait ouverte. Dans cette section, nous allons aller au-delà du jeu CHSH, pour cela nous allons prendre le point de vue des jeux non locaux. Dans ce cadre, pour analyser l'effet de l'incompatibilité des mesures d'Alice sur les effets non-locaux, nous fixons les mesures d'Alice. À partir de ses mesures, elle construira un tenseur et calculera la norme de G-Bell-(non)localité et la norme du tenseur de compatibilité. La nouvelle notion de G-Bell-(non)localité capture la violation d'une inégalité de Bell correspondant au jeu G. La norme de G-Bell-(non)localité est calculée en optimisant les mesures de Bob sur l'état quantique partagé. Nous disons que les

[1] French summary, as required by the doctoral school rules

mesures d'Alice sont G-Bell-locales si elles sont inférieures à la valeur classique du jeu, qui est l'espérance maximale de gagner le jeu dans le cadre classique. Nous avons montré, à l'aide de certaines inégalités, que la norme du tenseur de compatibilité et la norme de G-Bell-(non)localité ne sont, en général, pas égales. Cependant, dans [9], pour le jeu CHSH, les auteurs ont montré qu'elles sont équivalentes ; en traduisant ce résultat dans notre contexte, nous voyons que les deux normes sont égales. Avec l'équivalence forte au sens de [9], nous avons montré qu'avec des conditions suffisantes, le seul jeu satisfaisant cette équivalence est le jeu CHSH. Dans la Section 10.6, nous conclurons en passant en revue les contributions, et nous terminerons par quelques questions ouvertes et des directions de recherche futures.

10.1 Information quantique

Dans cette section, nous introduisons les concepts et outils de l'information quantique. *A un système physique $\mathcal{S}$, nous associons un espace de Hilbert complexe de dimension finie $\mathcal{H}$.* Pour des systèmes physiques plus compliqués comme l'oscillateur harmonique ou les systèmes de spin les espaces de Hilbert sont de dimension infinie.

Un état quantique est un vecteur normalisé $|\psi\rangle$ sur $\mathcal{H}$ qui encode toutes les *informations* du système physique. Deux vecteurs $|\psi\rangle$ et $|\varphi\rangle$ sur $\mathcal{H}$ décrivent le même système physique si

$$|\psi\rangle = \lambda |\varphi\rangle .$$

avec $\lambda \in \mathbb{C}^*$, les vecteurs sont définis modulo *une phase globale.*

Les observables en mécanique quantique caractérisant une quantité physique mesurable, comme par exemple l'énergie du système, sont décrites par des *opérateurs auto-adjoints* $A = A^* \in \mathcal{B}(\mathcal{H}) \cong \mathcal{M}_d(\mathbb{C})$. *En dimension finie, la décomposition spectrale d'une observable A est donnée par*

$$A = \sum_i \lambda_i P_i,$$

où $\lambda_i \in \mathbb{R}$ sont les valeurs propres de A et P_i sont les projecteurs[2]*. La distribution de probabilité d'obtenir le résultat λ_i lors de la mesure de l'observable A sur un système quantique dans un état $|\psi\rangle$ est donnée par la* règle de Born*:*

$$\mathbb{P}(\lambda_i) := \|P_i |\psi\rangle \|^2,$$

La mesure du résultat λ_i induit un changement de l'état quantique donné par

$$|\psi'\rangle := \frac{P_i |\psi\rangle}{\|P_i |\psi\rangle \|}.$$

Cet effet se nomme l'effondrement de la fonction d'onde.

L'évolution des systèmes quantiques est régie par une matrice unitaire $U \in \mathcal{U}(\mathcal{H})$: $|\psi'\rangle = U|\psi\rangle$. Un état initial $|\psi(t_0)\rangle$ évolue à un temps t vers un état $|\psi(t)\rangle$ ainsi on a

$$|\psi(t)\rangle := U(t, t_0) |\psi(t_0)\rangle .$$

La matrice unitaire $U(t, t_0)$, est donnée par

$$U(t, t_0) = e^{-iH(t-t_0)}.$$

où $H \in \mathcal{B}(\mathcal{H})$ est un opérateur auto-adjoint connu comme l'Hamiltonien du système quantique. En général, un système quantique est complètement décrit par des *matrices de densité* que nous

[2]Aussi les projecteurs sont connus dans la littérature sous le nom de mesure de von Neumann ou mesure à valeur projective qu'on notera par PVM.

désignerons par ρ. Les matrices de densité sont des *matrices définies positives de trace un.* Nous désignerons l'ensemble de ces matrices par $\mathcal{M}_d^{1,+}$ donné par:

$$\mathcal{M}_d^{1,+} := \Big\{\rho \in \mathcal{M}_d(\mathbb{C}) \,:\, \rho \geq 0\,;\, \mathrm{Tr}\,\rho = 1\Big\}.$$

L'ensemble des matrices de densité est un *ensemble convexe*, où les points extrémaux sont des projecteurs de rang un que nous noterons par $|\psi\rangle\langle\psi|$, où $\langle\psi|$ est le vecteur dual de $|\psi\rangle$.

Comme précédemment, l'évolution du système quantique est donnée par un unitaire $U \in \mathcal{U}(\mathcal{H})$, où l'évolution d'un état ρ_0 à t_0 vers ρ_t à t est donnée par:

$$\rho_t = U(t, t_0)\,\rho_0\,U^*(t, t_0).$$

La probabilité d'observer le résultat λ_i, la valeur propre d'une observable A est donnée par

$$\mathbb{P}(\lambda_i) = \mathrm{Tr}(P_i\,\rho) = \langle P_i\,,\rho\rangle_{HS}.$$

où nous avons utilisé le produit scalaire *Hilbert-Schmidt* sur $\mathcal{M}_d(\mathbb{C})$ définit comme suit :

$$\begin{aligned}\langle\cdot,\cdot\rangle_{HS} : \mathcal{M}_d(\mathbb{C}) \times \mathcal{M}_d(\mathbb{C}) \to \mathbb{C},&\\ (A, B) \to \langle A\,,B\rangle_{HS} := \mathrm{Tr}(A^*\,B).&\end{aligned}$$

La mesure induit un changement de l'état quantique, où l'état quantique résultant est donné par

$$\rho' = \frac{P_i\,\rho\,P_i}{\mathrm{Tr}(P_i\,\rho)}.$$

En fait, ces types de mesures sont très spécifiques et sont connus sous le nom de mesure projective, mais en général, le processus de mesure est décrit par *Positive operator Valued Measure* ou plus brièvement *POVM.* Les éléments des POVM sont des matrices positifs dont la somme est l'identité ainsi nous avons la définition suivante.

Definition 10.1.1. *Un POVM sur $\mathcal{M}_d$ avec k résultats possible est un k-tuple $A = (A_1, \ldots, A_k)$ d'opérateurs auto-adjoints de $\mathcal{M}_d$ qui definis positifs et dont la somme est égale à l'identité :*

$$\forall i \in [k]^3, \quad A_i \geq 0 \qquad et \qquad \sum_{i=1}^{k} A_i = I_d.$$

Lorsque l'on mesure un état quantique ρ avec l'appareil décrit par A, on obtient un résultat aléatoire de l'ensemble $[k]$:

$$\forall i \in [k], \qquad \mathbb{P}(outcome = i) = \mathrm{Tr}[\rho A_i].$$

Ce formalisme mathématique utilisé pour décrire les mesures quantiques (ou POVMs) ne rend pas compte de ce qui se passe avec la particule quantique après la mesure.

A fin de décrire un système physique $\mathcal{S}_{AB}$ composé de deux sous-systèmes $\mathcal{S}_A$ et $\mathcal{S}_B$ décrits par leurs espaces de Hilbert respectifs $\mathcal{H}_A \cong \mathbb{C}^{d_A}$ et $\mathcal{H}_B \cong \mathbb{C}^{d_B}$. Nous associerons au système physique $\mathcal{S}_{AB}$ l'espace de Hilbert total $\mathcal{H}_{AB}$ donné par le produit tensoriel de $\mathcal{H}_A$ et de $\mathcal{H}_B$ que nous noterons par $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Nous distinguerons deux classes d'états quantiques dans un système composite $\mathcal{S}_{AB}$, nous avons des états dit *intriqués* et d'autre *séparables*, plus précisément on a la définition suivante:

Definition 10.1.2. *Un état quantique $\rho_{AB} \in \mathcal{M}_{d_A d_B}^{1,+}$ est:*

- Produit, *si $\rho_{AB} = \rho_A \otimes \rho_B$.*

[3] Nous utilisons la notation $[k]$ pour l'ensemble $\{1, \cdots, k\}$.

- Séparable, *si c'est une* combinaison convexe d'états produits *:*

$$\rho_{AB} = \sum_x p_x \, \rho_A^x \otimes \rho_B^x$$

avec $p_x \geq 0$ *et* $\sum_x p_x = 1$.

- Intriqué, *si n'est pas séparable.*

Ci-dessus, d_A *et* d_B *sont les dimensions des espaces de Hilbert* $\mathcal{H}_A$ *et* $\mathcal{H}_B$ *respectivement.*

Nous introduisons les canaux quantiques, ces derniers sont des outil mathématique jouant un rôle fondamental dans la théorie de l'information quantique. D'un point de vue physique, ils représentent les opérations possibles que nous pouvons effectuer sur un système physique. D'un point de vue mathématique, ils représentent une classe d'application linéaire connues sous le nom d' *application complètement positives.*

Definition 10.1.3. *Une* application linéaire $\Phi(\cdot) : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_D(\mathbb{C})$ *est dite positive si elle satisfait la propriété suivante :*

$$X \in \mathcal{M}_d(\mathbb{C}) \quad , \quad X \geq 0 \implies \Phi(X) \geq 0.$$

Nous rappelons que $X \geq 0$ si $\sigma(X)$[4] $\subseteq [0, \infty[$. Une autre manière équivalente de définir $X \geq 0 \iff \exists Y \in \mathcal{M}_d(\mathbb{C})$, telle que $X = Y^*Y$. Une telle matrice X est appelée semi-définie positive.

Les *applications complétement positives*, peuvent être comprises comme une généralisation des applications positives.

Definition 10.1.4. *Une application linéaire* $\Phi(\cdot) : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_D(\mathbb{C})$ *est appelée* complètement positive *si pour tout* $K \geq 1$ *et* $X \in \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_K(\mathbb{C})$*, on a*

$$X \geq 0 \implies [\Phi \otimes \mathrm{id}_K](X) \geq 0,$$

où id_K *désigne la carte d'identité sur* $\mathcal{M}_K(\mathbb{C})$*. En d'autres termes,* Φ *est complètement positif si* $\forall K \geq 1$ *l'application* $\Phi \otimes \mathrm{Id}_K$ *est positive.*

Definition 10.1.5. *Une application complètement positive* $\Phi(\cdot)$*, est un canal quantique si elle préserve la trace :*

$$\forall Y \in \mathcal{M}_d(\mathbb{C}), \qquad \mathrm{Tr}\, \Phi(Y) = \mathrm{Tr}\, Y.$$

En fait, il existe une manière équivalente de décrire les canaux quantiques, qui ne nécessite pas l'utilisation de la structure du produit tensoriel.

Theorem 10.1.6. *Une application linéaire* $\Phi(\cdot) : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ *est un canal quantique si seulement si* $\exists \{L_i\}_{i=1}^k \subset \mathcal{M}_d(\mathbb{C})$ *satisfaisant les conditions suivantes :*

$$\forall X \quad , \quad \Phi(X) = \sum_{i=1}^k L_i X L_i^*.$$

et

$$\sum_{i=1}^k L_i^* L_i = I_d.$$

[4] Nous rappelons que $\sigma(X)$ est le *spectre de* X défini comme l'ensemble des nombres complexes z tels que $X - zI$ n'est pas inversible

10.2 Normes tensorielles

Dans cette section, nous introduisons les outils de normes tensorielles sur des espaces de Banach. Celle-ci joueront un rôle crucial pour comprendre le lien entre l'incompatibilité de mesures quantiques et la non-localité.

Nous rappelons qu'un *Espace de Banach* de dimension finie est un espace vectoriel X, doté d'une norme $\|\cdot\|_X$, que nous noterons par $(X, \|\cdot\|_X)$. Dans ce qui suit, nous utiliserons X, Y, Z à fin de désigner différents espaces de Banach avec leurs normes respectives $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$.

On note l'espace des applications linéaires de l'espace vectoriel X à Y par $\mathcal{L}(X, Y)$. En particulier, si nous considérons deux espaces de Banach $(X, \|\cdot\|_X)$ et $(Y, \|\cdot\|_Y)$, nous désignerons l'espace de toutes les applications linéaires de X à Y par $\mathcal{L}(X, Y)$ et nous avons

$$\varphi \in \mathcal{L}(X, Y) \quad , \quad \varphi : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y).$$

Remark 10.2.1. *Nous utiliserons la convention d'écrire $\mathcal{L}(X)$ au lieu de $\mathcal{L}(X, X)$. Rappelons que l'espace vectoriel dual X^* est identifié à $\mathcal{L}(X, \mathbb{C})$. Ses éléments sont connus sous le nom de formes linéaires.*

On peut mener l'espace des applications linéaires de X vers Y, d'une structure de norme définie par :

$$\varphi \in \mathcal{L}(X, Y) \quad , \quad \|\varphi\|_{X\to Y} := \sup\{ \|\varphi(x)\|_Y, \|x\|_X \leq 1\}.$$

Ainsi l'espace des applications linéaires est un espace de Banach muni de la norme définit au-dessus. En particulier, l'éspace des formes X^* est un éspace de Banach muni de la norme

$$\|\varphi\|_{X^*} := \sup\{ |\varphi(x)|, \|x\|_X \leq 1\}.$$

On note l'espace des formes linéaire muni de la norme définit au-dessus par $(X^*, \|\cdot\|_{X^*})$. Dans ce qui suit nous introduirons les espaces de Banach $\ell_{N(\mathbb{R})}$ et leurs analogues non commutatives les éspaces $\mathcal{S}_p^N(\mathbb{R})$ définit sur l'espace vectoriel des matrices, nous introduiserons aussi pour chaque cas l'énoncè du theorem de Hölder.

Definition 10.2.2. *L'espace de Banach $\ell_p^N(\mathbb{R})$ definit par $\ell_p^N(\mathbb{R}) = (\mathbb{R}^N, \|\cdot\|_p)$ avec la norme $\|\cdot\|_p$ est donnè par :*

$$\|\cdot\|_p : \mathbb{R}^N \to \mathbb{R}^+,$$

$$x \to \|x\|_p := \begin{cases} \Big(\sum_{i=1}^N |x_i|^p \Big)^{\frac{1}{p}} & if \quad 1 \leq p < \infty \\ \\ \sup\Big\{|x_i|, i \in \{1, \cdots, N\}\Big\} & if \quad p = \infty. \end{cases}$$

Theorem 10.2.3. *(Theorem de Hölder) Soit $x \in \ell_p^N(\mathbb{R})$ et $y \in \ell_q^N(\mathbb{R})$ avec des entiers p et q tels que $1 \leq p, q \leq +\infty$ verifiant la condition*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Alors on a

$$|\langle x, y\rangle| \leq \sum_{i=1}^N |x_i\, y_i| \leq \|x\|_p\, \|y\|_q.$$

Remark 10.2.4. *L'inégualitè ci-dessus ce réduit à* l'inigualitè de Cauchy-Schwarz *pour $p = q = 2$.*

Grâce au theorem de Hölder on peut identifier l'espace dual de $\ell_p^N(\mathbb{R})$ (voir [35]):

$$(\ell_p^N(\mathbb{R}))^* = (\mathbb{R}^N, \|\cdot\|_p)^* = \ell_q^N(\mathbb{R}) = (\mathbb{R}^N, \|\cdot\|_q),$$

avec p et q tels que $1 \leq p, q \leq +\infty$ et $\frac{1}{p} + \frac{1}{q} = 1$.

Soit $\mathcal{M}_N(\mathbb{C})$ l'espace vectoriel dess matrices, on peut munir l'espace des matrices par une norme connu sous le nom de *Schattern p−norm* ainsi on notera l'espace de Banach $\mathcal{S}_p^N(\mathbb{C}) := (\mathcal{M}_N(\mathbb{C}), \|\cdot\|_p)$.

Definition 10.2.5. *L'espace de Banach $\mathcal{S}_p^N(\mathbb{C}) = (\mathcal{M}_N(\mathbb{C}), \|\cdot\|_p)$ pour $1 \leq p \leq +\infty$,*

$$\|\cdot\|_p : \mathcal{M}_N(\mathbb{C}) \to \mathbb{R}^+,$$

$$M \to \|M\|_p := \begin{cases} \Big(\operatorname{Tr}|M|^p\Big)^{\frac{1}{p}} & si \quad 1 \leq p < \infty. \\ \\ \sup\Big\{\|M \cdot x\|,\ \|x\| \leq 1\Big\} & si \quad p = \infty. \end{cases}$$

avec $|M| := \sqrt{(M^*M)}$ et $\|\cdot\|$ est la norme euclidienne dans $\mathbb{C}^N$.

Ainsi dans le cadre non-commutative on a le *theorem de Hölder.*

Theorem 10.2.6. *(Theorem de Hölder) Soit $M \in \mathcal{S}_p^N(\mathbb{C})$ et $N \in \mathcal{S}_q^N(\mathbb{C})$ avec p et q de entiers tels que $1 \leq p, q \leq +\infty$ verifiant*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Alors on a

$$|\langle M, N\rangle| = |\operatorname{Tr}(M^* N)| \leq \|M\|_p \, \|N\|_q.$$

Remark 10.2.7. *L'inégalité ci-dessus ce réduit à* l'inégualité de Cauchy Schwarz *pour $p = q = 2$ avec le produit scalaire de* Hilbert-Schmidt *definit par $\langle M, N\rangle := \operatorname{Tr}(M^*N)$.*

Grâce au théoreme de Hölder, on peut identifier l'espace dual d'un espace de Banach non-commutative $\mathcal{S}_p^N(\mathbb{C})$. Ainsi on a la dualité suivante entre les espaces de Banach:

$$(\mathcal{S}_p^N(\mathbb{C}))^* = (\mathcal{M}_N(\mathbb{C}), \|\cdot\|_p)^* = \mathcal{S}_q^N(\mathbb{C}) = (\mathcal{M}_N(\mathbb{C}), \|\cdot\|_q),$$

avec p et q tels que $1 \leq p, q \leq \infty$ et $\frac{1}{p} + \frac{1}{q} = 1$.

Dans ce qui suit nous introduirons le concept de *norme tensorielle* sur des espaces de Banach, nous introduirons la norme tensoriel projective et injective qui joueront le rôle respectif d'une norme maximal et minimal qu'on peut obtenir lorsqu'on munit d'une norme l'espace tensoriel de deux espace de Banach.

On définit la *norme tensorielle projective* sur deux espaces de Banach $(X, \|\cdot\|_X)$ et $(Y, \|\cdot\|_Y)$.

Definition 10.2.8. *Soit deux espaces de Banch de dimension finie X et Y avec leurs normes respectives $\|\cdot\|_X$ et $\|\cdot\|_Y$. Soit $u \in X \otimes Y$, on définit la* norme tensorielle projective *de u par:*

$$\|u\|_{X \otimes_\pi Y} := \inf\left\{\sum_{i=1}^N \|x_i\|_X \|y_i\|_Y : u = \sum_{i=1}^N x_i \otimes y_i\right\},$$

avec l'infimum est pris sur toute les décompositions possible de $u = \sum_{i=1}^N x_i \otimes y_i$ avec N un entier arbitraire.

L'espace de Banach induit par la norme projective est $X \otimes_\pi Y := \Big(X \otimes Y, \|\cdot\|_{X \otimes_\pi Y}\Big)$.

Remark 10.2.9. *La définition de la norme projective peut être définie sur le produit tensoriel de plusieurs espaces. Afin d'illustrer cela, on considère M espaces de Banach $(X_i, \|\cdot\|_{X_i})$, avec $i \in \{1 \cdots M\}$. On munit le produit tensoriel des M Banach spaces par une norme projective définie par:*

$$u \in \bigotimes_{i=1}^{M} X_i \quad , \quad \|u\|_\pi := \inf \Big\{ \sum_{k=1}^{r} \|x_k^1\| \cdots \|x_k^M\| : r \in \mathbb{N} \, , \, x_k^i \in X_i \, , \, u = \sum_{k=1}^{r} x_k^1 \otimes \cdots \otimes x_k^M \Big\}.$$

où on a utilisé la notation $\|\cdot\|_\pi$ au lieu de $\|\cdot\|_{X_1 \otimes_\pi \cdots \otimes_\pi X_M}$.

La norme tensorielle projective satisfait *metric mapping property.*

Definition 10.2.10. *Soit les applications linéaires $T \in \mathcal{L}(X, Z)$ et $S \in \mathcal{L}(Y, W)$ où $(X, \|\cdot\|_X)\,,(Y, \|\cdot\|_Y)\,,(Z, \|\cdot\|_Z)\,,(W, \|\cdot\|_W)$ sont des espaces de Banch, on dit qu'une norme définie sur $X \otimes Y$ satisfait metric mapping property si pour tout application billineaire $T \otimes S$ on a:*

$$\|T \otimes S\| \leq \|T\| \, \|S\|.$$

Lemma 10.2.11. *Soit les applications linéaires $T \in \mathcal{L}(X, Z)$ and $S \in \mathcal{L}(Y, W)$, et les espaces de Banach $X \otimes_\pi Y$ et $Z \otimes_\pi W$. La norme tensorielle projective satisfait metric mapping property. Plus précisément on a:*

$$\|T \otimes S : X \otimes_\pi Y \to Z \otimes_\pi W\| \leq \|T\| \|S\|.$$

where $\|T \otimes S : X \otimes_\pi Y \to Z \otimes_\pi W\| := \sup_{\|u\|_{X \otimes_\pi Y} \leq 1} \|(T \otimes S)(u)\|$.

Dans ce qui suit, on introduit la *norme tensoriel injective* de deux espaces de Banach $(X, \|\cdot\|_X)$ et $(Y, \|\cdot\|_Y)$.

Definition 10.2.12. *Soit deux espaces de Banach de dimension finie X et Y avec leurs normes respectives $\|\cdot\|_X$ et $\|\cdot\|_Y$. Soit $u \in X \otimes Y$, on définit la* norme injective *de u par:*

$$\|u\|_{X \otimes_\varepsilon Y} := \sup_{\|\lambda\|_{X^*} \, , \, \|\sigma\|_{Y^*} \leq 1} |\langle \lambda \otimes \sigma, u \rangle|.$$

avec $\lambda \in X^*$ et $\sigma \in Y^*$ sont des formes linéaires.

Remark 10.2.13. *Dans la définition donné ci-dessus, on a utilisé l'abus de notation: $\langle \lambda \otimes \sigma, \cdot \rangle$ doit être comprise comme étant $(\lambda \otimes \sigma)(\cdot)$.*

L'espace de Banach induit par la norme injective est $X \otimes_\varepsilon Y := \Big(X \otimes Y, \|\cdot\|_{X \otimes_\varepsilon Y} \Big)$.

Remark 10.2.14. *La definition de la norme injective peut être definit pour plusieurs espaces de Banach. Afin d'illustrer cela, on considère M espaces de Banach $(X_i, \|\cdot\|_{X_i})$, avec $i \in \{1 \cdots M\}$. On munit le produit tensoriel des M espaces de Banach par la norme injective definit par:*

$$u \in \bigotimes_{i=1}^{M} X_i \quad , \quad \|u\|_\varepsilon := \sup \Big\{ |\langle x^1 \otimes \cdots \otimes x^M, u \rangle|; \, x^i \in X_i^* \, , \, \|x^i\|_{X_i^*} \leq 1 \Big\}.$$

où on a utilisé la notation $\|\cdot\|_\varepsilon$ au lieu de $\|\cdot\|_{X_1 \otimes_\varepsilon \cdots \otimes_\varepsilon X_M}$.

La norme tensorielle injective satisfait *metric mapping property.* L'introduction de la norme tensorielle projective et injective nous permet d'introduire la notion d'une *norme tensoriel* sur deux espaces de Banach $(X, \|\cdot\|_X)$ et $(Y, \|\cdot\|_Y)$.

Lemma 10.2.15. *Considérons la carte linéaire $T \in \mathcal{L}(X,Z)$ et $S \in \mathcal{L}(Y,W)$, ainsi que les espaces de Banach $X \otimes_\varepsilon Y$ et $Z \otimes_\varepsilon W$. La norme projective satisfait la metric mapping property. Explicitement on a :*

$$\|T \otimes S : X \otimes_\varepsilon Y \to Z \otimes_\varepsilon W\| \leq \|T\|\|S\|.$$

où $\|T \otimes S : X \otimes_\varepsilon Y \to Z \otimes_\varepsilon W\| := \sup_{\|u\|_{X\otimes_\varepsilon Y} \leq 1} \|(T \otimes S)(u)\|$.

Definition 10.2.16. *Soit $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ deux espaces de Banach et leurs espaces dual $(X^*, \|\cdot\|_{X^*})$ et $(Y^*, \|\cdot\|_{Y^*})$. On définit une norme α est une norme tensorielle sur $X \otimes Y$*

- *Pour $u = x \otimes y \in X \otimes Y$ on a:*

$$\|u\|_{X\otimes_\alpha Y} \leq \|x\|_X \, \|y\|_Y.$$

- *Pour tout $\varphi \in X^*$ et $\psi \in Y^*$, al forme bilineaire $\varphi \otimes \psi$ sur $X \otimes Y$ satisfait:*

$$\|\varphi \otimes \psi\|_{X^*\otimes_{\alpha^*} Y^*} \leq \|\varphi\|_{X^*} \, \|\psi\|_{Y^*}.$$

avec α^* est la norme dual de α définit sur $X^* \otimes Y^*$.

Remark 10.2.17. *On notera par $X \otimes_\alpha Y := (X \otimes Y, \|\cdot\|_{X\otimes_\alpha Y})$ l'espace de Banach induit par α sur $X \otimes Y$. La norme tensorielle dual α^* est une norme tensorielle sur $X^* \otimes Y^*$, l'espace de Banach induit par α^* est $X^* \otimes_{\alpha^*} Y^* := (X^* \otimes Y^*, \|\cdot\|_{X^*\otimes_{\alpha^*} Y^*})$. Ainsi on a la dualité suivante en dimension finie donné par:*

$$\Big(X \otimes_\alpha Y\Big)^* = X^* \otimes_{\alpha^*} Y^*.$$

Proposition 10.2.18. *Soit X et Y deux espaces de Banach.*

- *une norme α sur $X \otimes Y$ est une norme tensorielle si seulement si:*

$$\|u\|_{X\otimes_\varepsilon Y} \leq \|u\|_{X\otimes_\alpha Y} \leq \|u\|_{X\otimes_\pi Y}.$$

  pour tout $u \in X \otimes Y$.

- *Si α est une norme tensorielle sur $X \otimes Y$ alors $\|x \otimes y\|_{X\otimes_\alpha Y} = \|x\|_X \, \|y\|_Y$ pour tout $x \in X$ et $y \in Y$. Pour tout $\varphi \in X^*$ et $\psi \in Y^*$, la forme linaire $\varphi \otimes \psi$ sur $X \otimes_\alpha Y$ satisfait:*

$$\|\varphi \otimes \psi\|_{X^*\otimes_{\alpha^*} Y^*} = \|\varphi\|_{X^*} \, \|\psi\|_{Y^*}.$$

10.3 Non localité

Dans cette section, nous présentons le concept de non-localité dans le cadre des*jeux non-locaux* et leurs liens intrinsèques aux produits tensoriels des espaces de Banach.

Comme dans tout jeu, nous avons besoin de joueurs et de règles. Dans le cadre des jeux non locaux, les joueurs sont Alice et Bob et un *arbitre* dictant les règles du jeu. Avant le début du jeu, Alice et Bob sont autorisés à choisir une *stratégie* pour jouer le jeu, où la stratégie consiste à choisir l'un des différents ensembles de corrélation : soit classique, soit quantique, soit nonsignaling. Après leurs choix de stratégies ils se séparent et ne sont plus autorisés à communiquer. Lorsque le jeu commence, l'arbitre pose des questions aux joueurs, nous désignerons l'ensemble des questions pour Alice par $\mathcal{X}$ et pour Bob par $\mathcal{Y}$. L'arbitre choisit une paire de questions au hasard avec une distribution de probabilité $\pi : \mathcal{X} \times \mathcal{Y} \to \pi(x,y) \in [0,1]$ où $x \in \mathcal{X}$ désigne la question que l'arbitre envoie à Alice et $y \in \mathcal{Y}$ à Bob. Lorsque Alice et Bob reçoivent leurs questions, ils génèrent certaines réponses ou résultats $a \in \mathcal{A}$ et $b \in \mathcal{B}$. L'arbitre reçoit les réponses des joueurs et décide si elles sont correctes ou fausses, ce qui correspond au gain. Le

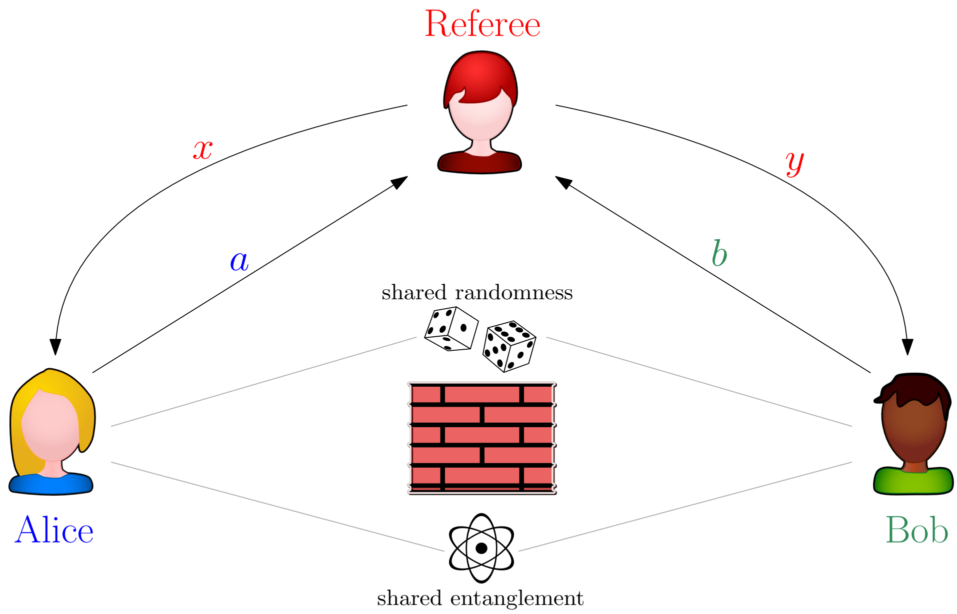


Figure 10.1: Représentation d'un jeu non-local, où Alice et Bob utilisent des stratégies classique ou quantique

gain est donné par $V : \mathcal{X} \times \mathcal{Y} \times \mathcal{A} \times \mathcal{B} \to V(x,y,a,b)$ où $V(x,y,a,b)$ peut être soit 0 soit 1 selon que les joueurs perdent ou gagnent la partie ; pour une illustration d'un jeu non local, voir la figure 10.1.

Le gain attendu d'Alice et de Bob est donné par:

$$\begin{aligned}\omega(G,\mathbb{P}) &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} \pi(x,y)\, V(x,y,a,b)\, \mathbb{P}(a\, b|x\, y) \\ &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} G^{a,b}_{x,y}\, \mathbb{P}(a\, b|x\, y).\end{aligned}$$

avec $G^{a,b}_{x,y} = \pi(x,y)\, V(x,y,a,b)$.

Le jeux CHSH est définit par $\mathcal{X} = \mathcal{Y} = \mathcal{A} = \mathcal{B} = \{\pm 1\}$. L'arbitre choisit uniformément de questions $\pi(x,y) = \frac{1}{4}$, pour tout x, y. Le gain du jeux CHSH est donné par

$$V(x,y,a,b) = \begin{cases} 1 & a \oplus b = x \cdot y \\ 0 & \text{autrement} \end{cases}$$

Definition 10.3.1. *On est maintenant en mesure de donner la valeur du jeux classique, quantique et nonsignaling.*

- *La* valeur classique d'un jeux $\omega(G)$ *est donnée par l'optimisation du gain sur l'ensemble de probabilités classique:*

$$\omega(G) = \sup_{\mathbb{P}_l(a\, b|x\, y)\in\mathcal{L}_{N,M}} |\omega(G,\mathbb{P}_l)|.$$

avec l'ensemble des probabilités classiques $\mathcal{L}_{N,M}$ donné par:

$$\mathcal{L}_{N,M} := \Big\{ \mathbb{P}_l(a\, b|x\, y) \,\Big|\, \mathbb{P}_l(a\, b|x\, y) = \int_\Lambda d\mu(\lambda)\, \mathbb{P}_{\mathbb{A}}(a|x\,,\lambda)\, \mathbb{P}_{\mathbb{B}}(b|y\,,\lambda) \Big\}.$$

- *La* valeur quantique d'un jeux $\omega^*(G)$ *est donnée par l'optimisation du gain sur l'ensemble de probabilités quantique:*

$$\omega^*(G) := \sup_{\mathbb{P}_Q(a\,b|x\,y)\in\mathcal{Q}_{N,M}} |\omega(G,\mathbb{P}_Q)|.$$

avec l'ensemble des probabilités quantiques $\mathcal{Q}_{N,M}$ *donné par:*

$$\mathcal{Q}_{N,M} := \Big\{\mathbb{P}_Q(a\,b|x\,y)\,\Big|\,\mathbb{P}_Q(a\,b|x\,y) = \langle\psi|A_{a|x}\otimes B_{b|y}\,|\psi\rangle\,;\,A_{a|x},B_{b|y}\geq 0,\ \forall x\sum_{a=1}^{M}A_{a|x} = I_{d_A};\forall y\sum_{b=1}^{M}B_{b|y} = I_{d_B}\Big\}.$$

- *La valeur* nonsignaling *d'un jeux* $\omega_{\mathcal{NS}}(G)$ *est donnée par l'optimisation du gain sur l'ensemble des probabilités de type nonsignaling:*

$$\omega_{\mathcal{NS}}(G) := \sup_{\mathbb{P}_{NS}(a\,b|x\,y)\in\mathcal{NS}_{N,M}} |\omega(G,\mathbb{P}_{NS})|.$$

avec l'ensemble des probabilités nonsignaling $\mathcal{NS}_{N,M}$ *donné par:*

$$\mathcal{NS}_{N,M} = \Big\{\mathbb{P}_{NS}(a\,b|x\,y)\Big|\sum_a\mathbb{P}_{NS}(a\,b|x\,y) = \sum_a\mathbb{P}_{NS}(a\,b|x'\,y)\,\forall b,x,x',y\ \textit{and}\sum_b\mathbb{P}_{NS}(a\,b|x\,y) = \sum_b\mathbb{P}_{NS}(a\,b|x\,y')\,\forall a,x,y',y\Big\}.$$

Dans ce qui suit, on se limitera dans le cadre des *jeux XOR* et on ne s'intéressera qu'aux stratégies classique et quantique. Un jeux XOR sont une classe de jeux où Alice et Bob donnent seulement deux réponses $\mathcal{A} = \mathcal{B} = \{0,1\}$ et le gain est donné par $V(x,y,a,b) := \frac{1}{2}(1+(-1)^{a\oplus b\oplus c_{x,y}})$ with $c_{x,y}\in\{0,1\}$. Ainsi dans le cadre des jeux de type XOR on a :

$$\begin{aligned}\omega(G,\mathbb{P}) &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}}\ \sum_{(a,b)\in\mathcal{A}\times\mathcal{B}}\pi(x,y)\,V(x,y,a,b)\,\mathbb{P}(a\,b|x\,y)\\ &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}}\ \sum_{(a,b)\in\{0,1\}^2}\pi(x,y)\frac{1}{2}(1+(-1)^{a\oplus b\oplus c_{xy}})\mathbb{P}(a\,b|x\,y)\\ &= \frac{1}{2}+\frac{1}{2}\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}}\pi(x,y)(-1)^{c_{x,y}}(\mathbb{P}(0\,0|x\,y)+\mathbb{P}(1\,1|x\,y)-\mathbb{P}(0\,1|x\,y)-\mathbb{P}(1\,0|x\,y))\\ &= \frac{1}{2}+\frac{\beta(G,\mathbb{P})}{2},\end{aligned}$$

où nous avons définit le *biais* d'un jeux XOR donné par

$$\beta(G,\mathbb{P}) := \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}}G_{x,y}\,(\mathbb{P}(0\,0|x\,y)+\mathbb{P}(1\,1|x\,y)-\mathbb{P}(0\,1|x\,y)-\mathbb{P}(1\,0|x\,y))\in[-1,1]$$

avec $G_{x,y} := \pi(x,y)(-1)^{c_{x,y}}$.

Definition 10.3.2. *Le* biais classique $\beta(G)$ *d'un jeux XOR est donné par:*

$$\beta(G) := \sup_{\gamma_{x,y}\in\mathbb{L}_N}\Big|\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}}G_{x,y}\,\gamma_{x,y}\Big|.$$

avec l'ensemble des corrélations classiques est donné par:

$$\mathbb{L}_N := \Big\{\gamma_{x,y}\Big|\gamma_{x,y} = \int_\Lambda A_x(\lambda)B_y(\lambda)\,\mathrm{d}\mu(\lambda);\,|A_x(\lambda)|,|B_y(\lambda)|\leq 1\Big\}\subseteq\mathcal{M}_N(\mathbb{R}).$$

où $A_x(\lambda) = \sum_{a\in\{0,1\}}a\,\mathbb{P}_{\mathbb{A}}(a|x,\lambda)$, $B_y(\lambda) = \sum_{b\in\{0,1\}}b\,\mathbb{P}_{\mathbb{B}}(b|y,\lambda)$.

Definition 10.3.3. *Le biais quantique $\beta^*(G)$ d'un jeux XOR est donné par:*

$$\beta^*(G) = \sup_{\gamma_{x,y}\in\mathbb{Q}_N} \Big| \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} G_{x,y}\, \gamma_{x,y} \Big|.$$

avec l'ensemble des corrélations quantiques est donné par:

$$\mathbb{Q}_N := \Big\{ \gamma_{x,y} \Big| \gamma_{x,y} = \langle\psi|\, A_x \otimes B_y \,|\psi\rangle\,;\ \|\psi\| = 1,\ \|A_x\|_\infty, \|B_y\|_\infty \leq 1 \Big\} \subseteq \mathcal{M}_N(\mathbb{R}).$$

où $A_x = A_{1|x} - A_{1|x}$ et $B_y = B_{1|y} - B_{1|y}$.

Le biais classique et quantique pour les jeux XOR sont intrinsèquement reliés aux normes tensorielles. A fin de mettre le lien en évidence en introduit une norme tensorielle γ_2.

Definition 10.3.4. *Soit deux espaces de Banach de dimension finie X et Y avec leurs normes respectives $\|\cdot\|_X$ et $\|\cdot\|_Y$. On définit la norme tensorielle γ_2 de $u \in X \otimes Y$ par:*

$$\|u\|_{X\otimes_{\gamma_2}Y} := \inf \left\{ \sup_{\alpha^*\in\mathbb{B}(X^*)} \left(\sum_{i=1}^N |\alpha^*(x_i)|^2 \right)^{\frac{1}{2}} \sup_{\beta^*\in\mathbb{B}(Y^*)} \left(\sum_{j=1}^N |\beta^*(y_j)|^2 \right)^{\frac{1}{2}} : u = \sum_{i=1}^N x_i \otimes y_i \right\}.$$

avec l'infimum est pris sur toute les décompositions possibles de $u = \sum_{i=1}^N x_i \otimes y_i$ avec $x_i \in X$ et $y_j \in Y$. On note $X \otimes_{\gamma_2} Y = (X \otimes Y, \|\cdot\|_{X\otimes_{\gamma_2}Y})$ induit par la norme γ_2.

Ainsi on peut relier le biais classique et quantique avec des normes tensorielles.

Theorem 10.3.5. *Le biais classique d'un jeux G est donné par la norme injective définie sur $\ell_1^N(\mathbb{R}) \otimes \ell_1^N(\mathbb{R})$, plus précisément on a:*

$$\beta(G) = \|G\|_{\ell_1^N(\mathbb{R})\otimes_\varepsilon \ell_1^N(\mathbb{R})}$$

ainsi par dualité l'ensemble de corrélations classiques est donné par:

$$\mathbb{L}_N = \mathbb{B}\Big(\ell_\infty^N(\mathbb{R}) \otimes_\pi \ell_\infty^N(\mathbb{R}) \Big).$$

avec $\mathbb{B}(X)$ est la boule unité d'un espace de Banach X.

Theorem 10.3.6. *Le biais quantique d'un jeux G est donné par la norme dual γ_2^* définie sur $\ell_1^N(\mathbb{R}) \otimes \ell_1^N(\mathbb{R})$, plus précisément on a:*

$$\beta^*(G) = \sup_{\gamma\in\mathbb{Q}} \{ |\langle G, \gamma\rangle| \} = \|G\|_{\ell_1^N(\mathbb{R})\otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})}$$

ainsi par dualité l'ensemble des corrélations quantiques est donné par:

$$\mathbb{Q}_N = \mathbb{B}\Big(\ell_\infty^N(\mathbb{R}) \otimes_{\gamma_2} \ell_\infty^N(\mathbb{R}) \Big),$$

avec la norme dual $\|G\|_{\ell_1^N(\mathbb{R})\otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})}$ est donné par:

$$\|G\|_{\ell_1^N(\mathbb{R})\otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})} := \sup \Big\{ |\langle G, \gamma\rangle| : \|\gamma\|_{\ell_\infty^N(\mathbb{R})\otimes_{\gamma_2} \ell_\infty^N(\mathbb{R})} \leq 1 \Big\}.$$

10.4 Compatibilité

L'une des principales différences entre le monde classique et le monde quantique est l'existence de mesures incompatibles qui décrivent des mesures que nous ne pouvons pas effectuer en même temps. Dans la section suivante, nous introduirons la notion d'(in)compatibilité des mesures quantiques, nous introduisons ça formulation en SDP. Nous introduirons la nouvelle notion de *compatibilité dimensionnelle* introduite dans [1]. Il se trouve qu'il existe certains types de mesures qu'on ne peu pas rendre compatible, à fin qu'il le devienne, on introduit l'effet d'un paramètre de bruit. Nous introduirons trois types de modèle de bruit et leurs lien avec le clonage asymétrique. Nous introduirons aussi la formulation de la compatibilité par une norme tensorielle, celle-ci jouera un rôle crucial pour comprendre le lien entre incompatibilité des mesures quantique et violation des inégalités de Bell.

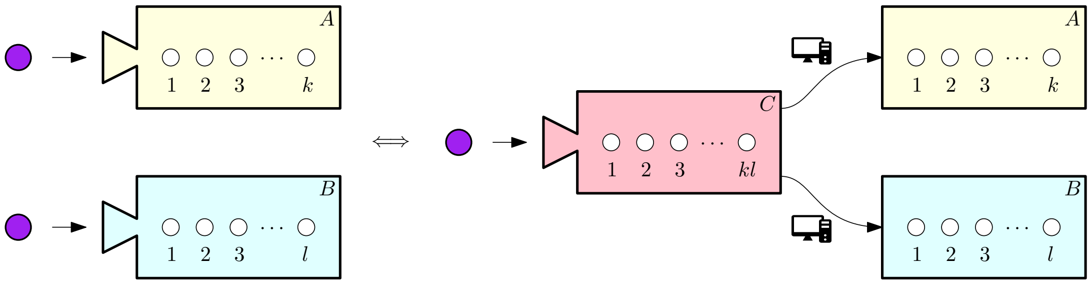


Figure 10.2: L'opérateur joint de A et de B peut être simulé par un troisième POVM C.

Definition 10.4.1. *Deux POVMs $A = (A_1, \dots, A_k)$, $B = (B_1, \dots, B_l)$ dans $\mathcal{M}_d$ sont dit* compatible *s'il existe un* POVM joint $C = (C_{11}, \dots, C_{kl})$ *dans $\mathcal{M}_d$ tels que A et B sont ses* marjinaux*:*

$$\forall i \in [k], \qquad A_i = \sum_{j=1}^{l} C_{ij}.$$

$$\forall j \in [l], \qquad B_j = \sum_{i=1}^{k} C_{ij}.$$

Plus généralement, a g-tuplets de POVMs $\mathbf{A} = (A^{(1)}, \dots, A^{(g)})$ sont compatible s'il existe un POVM C avec $[k_1] \times \cdots \times [k_g]$ résultats possibles tels que, pour tout $x \in [g]$, le POVM $A^{(x)}$ est le x-th marginal de C:

$$\forall i_x \in [k_x], \qquad A^{(x)}_{i_x} = \sum_{i_1=1}^{k_1} \cdots \sum_{i_{x-1}=1}^{k_{x-1}} \sum_{i_{x+1}=1}^{k_{x+1}} \cdots \sum_{i_g=1}^{k_g} C_{i_1 i_2 \cdots i_g}$$
$$= \sum_{\substack{\mathbf{j} \in [k_1] \times \cdots \times [k_g] \\ j_x = i_x}} C_{\mathbf{j}}.$$

La compatibilité de mesures quantiques est intrinsèquement différente de la commutativité, où le lien entre ces deux notions n'est possible que si on utilise des projecteurs. Ainsi plus précisément on a la proposition suivante:

Proposition 10.4.2. *Soit A_i et B_j deux observables dans un espace de Hilbert, si A_i et B_j satisfait l'inégalité suivante:*

$$\|[A_i, B_j]\| \leq 4\|A_i - A_i^2\| \cdot \|B_j - B_j^2\|.$$

alors A_i, et B_j sont compatibles.

En particulier si A_i et B_j sont des PVMs alors ils sont compatibles si et seulement si

$$[A_i, B_j] = 0. \quad \forall i, j.$$

On peut illustrer l'opérateur joint C comme une grande boîte noire où on peut déduire à la fois les mesures A et B (voir Figure 10.2 pour une illustration).

Proposition 10.4.3. *Soit un N-uplets de POVMs $\mathbf{A} = (A^{(1)}, \dots, A^{(N)})$. Ces derniers sont compatibles si et seulement si il existe POVM joint $(C_k)_{k\in[K]}$ et une famille de probabilités conditionnelles $\big(p_x(\cdot|\cdot)\big)_{x\in[N]}$ tels que*

$$\forall x \in [N], \forall i \in [k_x], \qquad A_i^{(x)} = \sum_{k\in[K]} p_x(i|k) C_k.$$

Déterminer si des POVMs sont compatibles est généralement très compliqué, par ailleurs il existe une méthode numérique permettant de déterminer s'ils sont compatibles connue sous le nom de *programme semidefinit* ou simplement SDP.

Proposition 10.4.4. *Soit deux POVMs $\{Q, I - Q\}$ et $\{P, I - P\}$, avec P, Q sont des matrices autoadjointes de dimension $d \times d$ tels que $0 \leq P, Q \leq I_d$. Les deux POVMs sont compatible si seulement si $\varepsilon_0 \leq 1$, avec*

$$\varepsilon_0 := \inf\big\{\varepsilon \,:\, \exists \delta \geq 0 \quad s.t. \quad \delta + I - Q - P \geq 0,\, Q + \varepsilon I - \delta \geq 0,\, P + \varepsilon I - \delta \geq 0\big\}, \tag{10.1}$$

où X est une matrice définie positive.

Proposition 10.4.5. *La formulation duale du programme semi-définie est donnée par:*

$$\varepsilon^* = \sup_{X,Y,Z\geq 0}\left\{ \mathrm{Tr}[X(Q+P-I)] - \mathrm{Tr}[YQ] - \mathrm{Tr}[PZ] \; with \, X \leq Y + Z,\, \mathrm{Tr}[Y+Z] = 1\right\},$$

L'introduction de la compatibilité dimensionnelle a été le résultat majeur de l'article [1], on a pu analyser l'effet de la dimension de l'espace de Hilbert sur la compatibilité des mesures quantiques.

Definition 10.4.6. *Soit un g-uplets de POVMs $\mathbf{A} = (A^{(1)}, \dots, A^{(g)})$ défini sur un espace de dimension d, on définit la notion de* compatibilité dimensionnel *comme étant la dimension maximale r pour laquelle il* existe *une isométrie $V : \mathbb{C}^r \to \mathbb{C}^d$ telle que les g-uplets de POVMs deviennent compatibles :*

$$R(\mathbf{A}) := \max\{r \in [d] \,:\, \exists V : \mathbb{C}^r \to \mathbb{C}^d \text{ isom. s.t. } V^* A^{(1)} V, \dots, V^* A^{(g)} V \text{ sont comp.}\}$$

On définit la notion forte de compatibilité dimensionnelle *si pour un g-uplets de POVMs $\mathbf{A}$, la dimension maximale r telle que pour* toute *isometrie $V : \mathbb{C}^r \to \mathbb{C}^d$ le g-tuple POVM soit compatible:*

$$\bar{R}(\mathbf{A}) := \max\{r \in [d] \,:\, \forall V : \mathbb{C}^r \to \mathbb{C}^d \text{ isom., } V^* A^{(1)} V, \dots, V^* A^{(g)} V \text{ sont comp.}\}$$

Dans ce qui suit, nous introduirons les différents types de modèles de bruit et leurs liens avec le clonage asymétrique et on donnera une application de compatibilité dimensionnelle dans le cadre des projecteurs bruités construits à partir des MUB.

Il existe une procédure qui rend les mesures incompatibles aussi compatibles que nous le souhaitons. Ceci est réalisé en ajoutant un certain *bruit* classique aux POVMs. Le bruit est généralement donné par un paramètre $t \in [0, 1]$ qui mélangera les POVMs originaux avec un *opérateur de mesure trivial.* Intuitivement, plus le paramètre t croît, plus les POVMs deviennent compatibles. Dans ce qui suit, nous présenterons différents *modèles de bruit* qui ont été établis. Nous présenterons également le lien entre l'un des modèles de bruit et le problème du clonage asymétrique, qui est un moyen de contourner le théorème de non-clonage. Le premier type de modèle de bruit que nous allons introduire est le *bruit blanc.* Au lieu de mesurer la POVM $A_1, \cdots, A_N$, on mesure la POVM bruyante $A'_1, \cdots, A'_N$ donnée par une combinaison convexe de $A_1, \cdots, A_N$ et du I_d avec un paramètre $t \in [0,1]$:

$$A_i \to A'_i := t\, A_i + (1-t)\frac{I_d}{N}, \quad i \in [N].$$

Le nouveau POVM $(A'_1, \cdots, A'_N)$ correspond à un dispositif qui effectue la mesure originale avec une probabilité t et qui, avec une probabilité $1-t$, produit un résultat uniformément aléatoire. En fait, la POVM $\mathcal{I} := (\frac{I_d}{N}, \cdots, \frac{I_d}{N})$ est une POVM triviale : les mesures de $\mathcal{I}$ produisent les mêmes statistiques de résultat pour chaque état quantique. Il existe d'autres classes d'opérateurs triviaux où la POVM $\mathcal{I}$ est une POVM spéciale. La classe des POVM triviaux est de la forme $E = (e_1 I_d, \cdots, e_N\, I_d)$ avec $e := (e_1, \cdots, e_N)$ est une distribution de probabilité, où le choix d'une distribution donnée spécifie complètement le type de modèle de bruit. Un autre type de modèle de bruit qui a également été considéré dans la littérature est donné par $e := (\mathrm{Tr}[A_1]/d \cdot I_d, \cdots, \mathrm{Tr}[A_N]/d \cdot I_d)$ qui dépend du POVM initial lui-même

$$A_i \to A'_i := t\, A_i + (1-t)\frac{\mathrm{Tr}[A_i]}{d}\, I_d, \quad i \in [N].$$

On voit immédiatement que ce modèle de bruit est linéaire en A_i.

Remark 10.4.7. *Dans toute cette description, nous n'avons considéré qu'un seul paramètre de bruit t. On peut en fait étudier ces différents types de modèles de bruit avec un vecteur de paramètres $\mathbf{t} := (t_1, \cdots, t_g) \in [0,1]^g$ en clonant g-tuples de POVMs*

Dans ce qui suit, nous allons introduire la connexion entre les modèles de bruit et le *clonage quantique approximatif.* Le théorème de non-clonage est l'un des concepts clés qui différencient le monde classique du monde quantique, techniquement on ne peut pas construire un canal quantique $\Phi : \mathcal{M}_d \to \mathcal{M}_d^{\otimes g}$ tel que pour tout

$$\rho \in \mathcal{M}_d^{1,+}, \forall j \in \{1, \cdots, g\}, \qquad \mathrm{Tr}_{[g]\setminus\{j\}}\, \Phi(\rho) = \rho.$$

Le clonage quantique approximatif (symétrique ou asymétrique) caractérise une machine à cloner imparfaite, dont le rôle est de produire des clones (copies) imparfaits pour des états quantiques d'entrée arbitraires. L'imperfection repose sur le fait que nous agissons avec le canal quantique, et en prenant les marginales du canal nous obtenons un état résiduel bruyant décrit par une combinaison convexe de l'état initial et d'un opérateur trivial avec un certain paramètre $t \in [0,1]$. La machine de clonage quantique asymétrique est caractérisée par un uplet de paramètres $t_i \in [0,1]^g$ et le cas symétrique se réduit à un seul paramètre $t \in [0,1]$. Formellement, nous avons la définition suivante du clonage quantique asymétrique :

Definition 10.4.8. *L'ensemble des paramètres correspondant à un clonage asymétrique $1 \to g$ dans $\mathbb{C}^d$ est décrit par l'ensemble suivant:*

$$\Gamma^{clone}(g,d) := \Big\{ \mathbf{t} \in [0,1]^g \ : \ \exists \Phi : \mathcal{M}_d \to \mathcal{M}_d^{\otimes g} \text{ canal quantique tel que } \forall \rho \in \mathcal{M}_d, \forall j \in [g], \quad \mathrm{Tr}_{[g]\setminus\{j\}}\, \Phi(\rho) = t_j \rho + (1-t_j)\frac{I_d}{d} \Big\}.$$

Le clonage des états quantiques peut être réinterprétée dans la représentation de de Heisenberg de la mécanique quantique en regardant l'application duale d'un canal ; cette opération agit naturellement sur les mesures quantiques. Cette dualité permet de faire le lien entre l'obtention des clones imparfaits et le fait d'utiliser des mesures bruités. Définissons l'application duale correspondant de paramètres de clonage. Considérons l'ensemble de paramètres associé à l'application duale donné par:

$$\tilde{\Gamma}^{clone}(g,d) := \Big\{ \mathbf{t} \in [0,1]^g \,:\, \exists \Psi : \mathcal{M}_d^{\otimes g} \to \mathcal{M}_d \text{ unital and completely positive such that} \\ \forall X \in \mathcal{M}_d, \forall j \in [g], \quad \Psi(I^{\otimes(j-1)} \otimes X \otimes I^{\otimes(g-j)}) = t_j X + (1-t_j)\frac{\operatorname{Tr} X}{d} I \Big\}.$$

Proposition 10.4.9. *L'espace des paramètres de clonages asymétriques et son dual sont identiques:* $\forall g, d \geq 2$,

$$\tilde{\Gamma}^{clone}(g,d) = \Gamma^{clone}(g,d).$$

Theorem 10.4.10. *Soit* $\mathbf{A} = (A^{(1)}, \ldots, A^{(g)}$ *un g-uplets de POVM dans* $\mathcal{M}_d$ *ayant* $k_1, \ldots, k_g$ *résultats. Soit pour tout* $x \in [g]$,

$$s_x := 1 - \min_{i \in [k_x]} \frac{d\lambda_{\min}(A_i^{(x)})}{\operatorname{Tr} A_i^{(x)}} \in [0,1].$$

avec $\lambda_{min}(\cdot)$ *la valeur propre minimale d'un opérateur. Si* $\mathbf{s} \in \Gamma^{clone}(g,d)$*, alors les POVMs* $\mathbf{A}$ *sont compatibles.*

Dans ce qui suit une application de la compatibilité dimensionnelle: soit de POVMs construits à partir des MUBs, on peut trouver un espace de Hilbert de dimension plus petite tel que ces dernier deviennent compatibles. Pour cela on se donne un g uplets de bases orthonormales $\Big\{ \{|b_i^{(x)}\rangle\}_{i \in [d]} \Big\}_{x \in [g]}$ on dit qu'elles sont *mutually unbiased* (MUB) si

$$\forall x \neq y \in [g], \forall i,j \in [d], \qquad |\langle b_i^{(x)} | b_j^{(y)} \rangle|^2 = \frac{1}{d}.$$

Soit deux mutually unbiased bases $\{|a_1\rangle, \ldots, |a_d\rangle\}$ and $\{|b_1\rangle, \ldots, |b_d\rangle\}$ in $\mathbb{C}^d$. Soit la version bruités de A et B donné par:

$$\mathcal{N}_\lambda[A] = \left(\lambda |a_1\rangle\langle a_1| + (1-\lambda)\frac{I_d}{d}, \ldots, \lambda |a_d\rangle\langle a_d| + (1-\lambda)\frac{I_d}{d} \right)$$
$$\mathcal{N}_\mu[B] = \left(\mu |b_1\rangle\langle b_1| + (1-\mu)\frac{I_d}{d}, \ldots, \mu |b_d\rangle\langle b_d| + (1-\mu)\frac{I_d}{d} \right).$$

Pour $(\lambda, \mu) \in [0,1]^2$, $\mathcal{N}_\lambda[A]$ and $\mathcal{N}_\mu[B]$ sont compatibles si et seulement si

$$\lambda + \mu \leq 1 \text{ ou } \lambda^2 + \mu^2 + \frac{2(d-2)}{d}(1-\lambda)(1-\mu) \leq 1.$$

Pour le cas symétrique où $\lambda = \mu$, les POVMs $\mathcal{N}_\lambda[A]$ et $\mathcal{N}_\lambda[B]$ sont compatibles si seulement si:

$$\lambda \leq \frac{1}{2}\left(1 + \frac{1}{1+\sqrt{d}}\right).$$

Comme application de la compatibilité dimensionnelle on a le théorème suivant où même si on considère que des PVMs bruités construits à partir de MUB celle restent incompatible alors il existe une dimension $r < \sqrt{d}$ tels qu'ils deviennent compatible.

Theorem 10.4.11. *Soit deux POVMs* A, B *correspendant à des mutually unbiased bases, celle-ci pouvant être etendu à un triplet de MUBs. Pour tout* $2 \leq r < \sqrt{d}$*, il existe un interval non vide* $\Lambda_{r,d} \subset [0,1]$ *tel que, pour tout* $\lambda \in \Lambda_{r,d}$,

- *les mesures bruitès de types MUB $\mathcal{N}_\lambda[A]$, $\mathcal{N}_\lambda[B]$ sont incompatible*
- *leurs version reduite $V^*\mathcal{N}_\lambda[A]V$, $V^*\mathcal{N}_\lambda[B]V$ sont compatible,*

avec $V : \mathbb{C}^r \to \mathbb{C}^d$ est une isométrie obtenu à partir d'un troisième MUB où l'isométrie $V := \sum_{k=1}^r |c_k\rangle\langle k|$ et

$$\Lambda_{r,d} := \left(\frac{2+\sqrt{d}}{2(1+\sqrt{d})}, \frac{2+r}{2(1+r)}\right]$$

Dans la suite de cette section nous introduisons la nouvelle description de la compatibilité par une norme tensorielle $\|\cdot\|_c$ qu'on nommera *norme de compatibilité*. Celle-ci jouera un rôle crucial à la compréhension du lien entre l'incompatibilité des mesures quantiques et la violation des inégalités de Bell.

Definition 10.4.12. *Soit un tenseur $A \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$, on défit la quantité suivante:*

$$\|A\|_c := \inf\left\{\Big\|\sum_{j=1}^K H_j\Big\|_\infty \;:\; A = \sum_{j=1}^K z_j \otimes H_j,\;\; s.t.\;\; \forall j \in [K],\, \|z_j\|_\infty \le 1 \;\; et \;\; H_j \ge 0\right\}.$$

La quantité $\|\cdot\|_c$ est une norme tensorielle sur $(\mathbb{R}^N, \|\cdot\|_\infty)\otimes(\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$, plus précisément on a:

$$\|A\|_{\ell_\infty^N(\mathbb{R})\otimes_\varepsilon \mathcal{S}_\infty^d(\mathbb{C})} \le \|A\|_c \le \|A\|_{\ell_\infty^N(\mathbb{R})\otimes_\pi \mathcal{S}_\infty^d(\mathbb{C})}$$

avec la norme injective et projective définit sur $\ell_\infty^N(\mathbb{R}) \otimes \mathcal{S}_\infty^d(\mathbb{C})$ are sont donné par:

$$\|A\|_{\ell_\infty^N(\mathbb{R})\otimes_\varepsilon \mathcal{S}_\infty^d(\mathbb{C})} := \sup\Big\{\langle x\otimes Y, A\rangle, \|x\|_{\ell_1^N(\mathbb{R})} \le 1, \|Y\|_{\mathcal{S}_1^d(\mathbb{C})} \le 1\Big\}.$$

et

$$\|A\|_{\ell_\infty^N(\mathbb{R})\otimes_\pi \mathcal{S}_\infty^d(\mathbb{C})} := \inf\Big\{\sum_i \|x_i\|_{\ell_1^N(\mathbb{R})}\|Y_i\|_{\mathcal{S}_1^d(\mathbb{C})};\, A = \sum_i x_i \otimes Y_i\Big\}.$$

plus précisément on a la proposition suivante

Proposition 10.4.13. *La quantité $\|\cdot\|_c$ est une norme tensorielle sur $(\mathbb{R}^N, \|\cdot\|_\infty)\otimes(\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$.*

La norme tensorielle $\|\cdot\|_c$ caractérise si les mesures avec deux résultats possibles sont compatibles. Le théorème suivant donne une formulation géométrique de la compatibilité.

Theorem 10.4.14. *Soit $A = (A_1, \ldots, A_N)$ est un N-uplets de matrices complexe auto-adjointe de dimension $d \times d$. Alors on a:*

1. *A est une collection de mesures dichotomiques d'observable quantique (i.e. $\|A_i\|_\infty \le 1\ \forall i$) si seulement si $\|A\|_\varepsilon \le 1$, où $\|\cdot\|_\varepsilon$ est une norme tensorielle sur $\ell_\infty^N \otimes_\varepsilon S_\infty^d$.*
2. *A est une collection de mesure dichotomique* compatible *si seulement si $\|A\|_c \le 1$.*

10.5 Comptabilité et non-localité

Dans cette section, nous donnerons un cadre commun pour analyser l'incompatibilité des mesures et la non-localité. Pour cela, nous considérerons que Alice et Bob jouent un jeu non local, où les mesures d'Alice sont fixes. Si ses mesures sont incompatibles, elle veut savoir si elle viole une inégalité de Bell. Pour cela, elle calculera deux normes tensorielles $\|\cdot\|_c$ et $\|\cdot\|_G$ d'un tenseur construit à partir de ses dispositifs de mesure. La compréhension du lien entre incompatibilité et non-localité se traduit dans ce cadre par la comparaison des deux normes $\|\cdot\|_c$ et

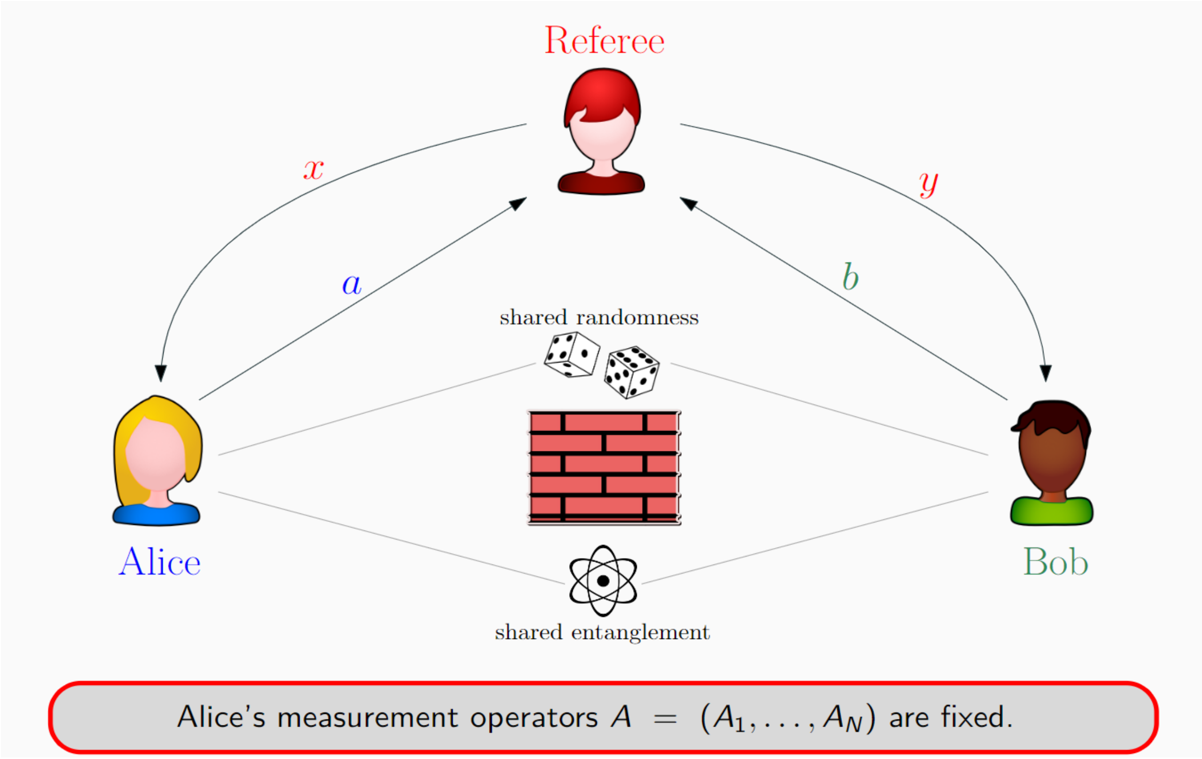


Figure 10.3: Alice et Bob jouent le jeu XOR avec les mesures d'Alice sont fixées.

$\|\cdot\|_G$. Le résultat obtenu dans [9], est que pour le jeu CHSH, l'incompatibilité est équivalente à la violation de l'inégalité de Bell, ce qui signifie dans notre cadre que $\|\cdot\|_{G_{CHSH}} = \|\cdot\|_c$. La question est restée ouverte pour les jeux généraux G. A fin d'unifier l'incompatibilité des mesures quantiques et les violations de l'inégalité de Bell, nous utiliserons le cadre naturel des jeux non locaux et en particulier les jeux XOR généralisant celui de CHSH. Mais au lieu d'optimiser la mesure d'Alice à travers le biais quantique, nous supposerons dans ce cadre que *les mesures d'Alice sont fixes*. Pour un jeu G donné, elle demande *si elle viole une inégalité de Bell avec des mesures incompatibles* (voir figure 10.3 pour une représentation de l'expérience de pensée). Pour unifier ces deux notions fondamentales de la théorie quantique, l'*incompatibilité des mesures* et la *violation des inégalités de Bell*, nous considérerons le cadre des *jeux XOR non locaux*, où les règles du jeu sont encodées dans une matrice réelle G de taille $N \times N$, et *les mesures dichotomiques d'Alice sont fixées.*

Comme on l'avait décrit auparavant un jeu nonlocal est complètement décrit par une matrice réelle G de dimension $N \times N$. Si Alice veut savoir si elle viole une inégalité de Bell, elle doit calculer la norme suivante $\|A\|_G$ associée à un N-uplets de ces appareils de mesure donné par $A = (A_1, \cdots, A_N) \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$ où $A_x = A_{0|x} - A_{1|x}$ est l'observable correspondant au POVM dichotomique $(A_{0|x}, A_{1|x})$.

Definition 10.5.1. *Soit un jeu non local décrit par la matrice $G \in \mathcal{M}_N(\mathbb{R})$. Les mesures d'Alice étant fixées, elles sont décrites par un N-uplets d'observable $A = (A_1, \ldots, A_N) \in \mathcal{M}_d^{sa}(\mathbb{C})^N$. La valeur maximale du biais quantique du jeu G avec des mesures d'Alice fixe décrit par l'observable A_x est donnée par:*

$$\|A\|_G := \sup_{\|\psi\|=1} \sup_{\|B_y\|\leq 1} \left\langle \psi \middle| \sum_{x,y=1}^{N} G_{xy} A_x \otimes B_y \middle| \psi \right\rangle,$$

où le supremum est pris sur un état bipartite $\psi \in \mathbb{C}^d \otimes \mathbb{C}^D$ et les mesures de Bob $B = (B_1, \ldots, B_N) \in \mathcal{M}_D^{sa}(\mathbb{C})^N$, avec D un paramètre de dimension libre.

Definition 10.5.2. *Soit un jeu non local G, on dit que les mesures d'Alice $A = (A_1, \dots, A_N)$ sont G-*Bell-locale *si pour tout choix des mesures de Bob B et pour tout état quantique partagé ψ, on ne peut violer une inégalité de Bell correspondant au jeux G:*

$$\|A\|_G \leq \beta(G).$$

*si l'inégalité n'est pas satisfaite on dit que les mesures d'Alice sont G-*Bell-non-locales.

L'intuition physique de la définition donnée ci-dessus, est pour n'importe qu'elle optimisation sur les mesures de Bob et pour toute optimisation sur les états quantiques partagés, si Alice n'arrive pas à obtenir des résultats supérieurs au biais classique alors ces mesures sont locales.

Lemma 10.5.3. *Soit un jeu quantique $(G_{xy})_{\{x,y=1\}}^N$ on peut donner une formulation équivalente à $\|A\|_G$ donné par:*

$$\|A\|_G = \lambda_{\max}\left[\sum_{y=1}^N \left|\sum_{x=1}^N G_{xy}\, A_x\right|\right].$$

Remark 10.5.4. *Dans la définition de $\|\cdot\|_G$ la dimension de l'espace des mesures d'Alice est fixé à (d), par ailleurs la dimension de l'espace de Hilbert de Bob est libre et donné par (D). Dans ce qui suit, nous montrerons qu'en toute généralité il suffit de prendre les dimensions des espaces de Hilbert d'Alice et de Bob comme étant les mêmes, on montrera que cela est suffisant pour le problème d'optimisation. Assumant que $D \geq d$, un état quantique $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^D$, et N opérateur de mesure à deux résultats $B_1, \dots, B_N \in \mathcal{M}_D^{sa}(\mathbb{C})$. En utilisant la décomposition de Schmidt sur un état pure bipartite $|\psi\rangle$ induira une réduction effective de la dimension de l'espace de Hilbert de Bob de D à d. Soit la décomposition de Schmidt de $|\psi\rangle$ donné par:*

$$|\psi\rangle = \sum_{i=1}^d \sqrt{\lambda_i}\, |a_i\rangle \otimes |b_i\rangle.$$

Dans l'équation donnée, notez que le nombre de termes est borné par la dimension la plus petite entre d et D donc d. La décomposition orthonormale de la famille $\{|b_i\rangle\}_{i\in[d]}$ engendre le sous-espace de dimension d dans $\mathbb{C}^D$. Soit une une base orthonormale *$\{|\tilde{b}_i\rangle\}_{i\in[d]}$ de $\mathbb{C}^d$ et une isométrie*

$$V : \mathbb{C}^d \to \mathbb{C}^D \quad \text{tel que} \quad \forall i \in [d], \quad V\,|\tilde{b}_i\rangle = |b_i\rangle.$$

En introduisant l'état quantique donné par:

$$\mathbb{C}^d \otimes \mathbb{C}^d \ni |\tilde{\psi}\rangle := \sum_{i=1}^d \sqrt{\lambda_i}\, |a_i\rangle \otimes |\tilde{b}_i\rangle$$

Ainsi les appareils de mesures sont donné par:

$$\mathcal{M}_d^{sa}(\mathbb{C}) \ni \tilde{B}_y := V^* B_y V, \quad \forall y \in [N].$$

La normalisation de l'état quantique et la réduction des $\tilde{B}_y$ sont dûs à l'isométrie V. Ainsi on a

$$\begin{aligned}
\left\langle \psi \middle| \sum_{x,y=1}^N G_{xy}\, A_x \otimes B_y \middle| \psi \right\rangle &= \sum_{x,y=1}^N G_{xy} \sum_{i,j=1}^d \sqrt{\lambda_i \lambda_j} \langle a_i | A_x | a_j \rangle \underbrace{\langle b_i | B_y | b_j \rangle}_{=\langle \tilde{b}_i | V^* B_y V | \tilde{b}_j \rangle} \\
&= \sum_{x,y=1}^N G_{xy} \sum_{i,j=1}^d \sqrt{\lambda_i \lambda_j} \langle a_i | A_x | a_j \rangle \langle \tilde{b}_i | \tilde{B}_y | \tilde{b}_j \rangle \\
&= \left\langle \tilde{\psi} \middle| \sum_{x,y=1}^N G_{xy}\, A_x \otimes \tilde{B}_y \middle| \tilde{\psi} \right\rangle.
\end{aligned}$$

Ainsi on a pu montrer que toute les corrélations qu'on peut obtenir avec les mesures de Bob sur un espace de dimension D, sont les même corrélations qui peuvent être obtenues si les dimensions des espaces de Hilbert d'Alice et de Bob sont égales.

L'un des résultats majeurs de cette section est que $\|A\|_G$ est une norme tensorielle, plus précisément on a:

$$\|A\|_{\mathbb{R}^N \otimes_\varepsilon \mathcal{M}_d^{sa}(\mathbb{C})} \leq \|A\|_G \leq \|A\|_{\mathbb{R}^N \otimes_\pi \mathcal{M}_d^{sa}(\mathbb{C})}$$

avec $(\mathbb{R}^N, \|\cdot\|_G)$ and $(\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$.

On munit les espaces vectoriels $\mathbb{R}^N$ et $\mathcal{M}_d^{sa}(\mathbb{C})$ avec leurs normes respectives données par $\|\cdot\|_G$ et la norme d'opérateur (ou la norme Schatten-∞, $\mathcal{S}_\infty$). Notez qu'on a utilisé l'abus de notation suivant: on a utilisé $\|\cdot\|_G$ pour une norme dan $\mathbb{R}^N$ et dans $\mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$; cet abus de notation sera clair à partir du contexte. Nous allons explorer les propriétés de la norme $\|\cdot\|_G$ vis à vis du produit tensoriel. Soit un N-uplets d'observables $(A_1, A_2, \ldots, A_N)$, leur associant la quantité suivante:

$$A := \sum_{x=1}^{N} e_x \otimes A_x \in \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C}).$$

Definition 10.5.5. *Soit $p \in \mathbb{R}^N$, on définit la quantité suivante:*

$$\|p\|_G := \sum_{y=1}^{N} \left| \sum_{x=1}^{N} G_{xy}\, p_x \right| = \|G^\top p\|_1.$$

Lemma 10.5.6. *Soit une matrice inversible G, la fonction $\mathbb{R}^N \ni p \mapsto \|p\|_G$ est une norme.*

Dans ce qui suit on étudiera l'espace dual de $(\mathbb{R}^N, \|\cdot\|_G)$. pour cela on calculera la norme duale de $\|\cdot\|_G$ donnée par $\|\cdot\|_G^*$.

Proposition 10.5.7. *La norme duale $\|\cdot\|_G^*$ est donnée par :*

$$\forall p \in \mathbb{R}^N, \qquad \|p\|_G^* = \max_y \left| \sum_{z=1}^{N} (G^{-1})_{yz}\, p_z \right| = \|G^{-1} p\|_\infty.$$

Le théorème suivant montre que $\|\cdot\|_G$ est une norme tensorielle dans $\mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$.

Theorem 10.5.8. *Soit un jeu non local inversible G de N questions et réponses, la quantité $\|\cdot\|_G$ caractérisant la valeur maximale du biais quantique d'un jeu G avec les mesures dichotomique d'Alice fixe, est une norme tensorielle dans $\mathcal{M}_d^{sa}(\mathbb{C})^N \cong \mathbb{R}^N \otimes \mathcal{M}_d^{sa}(\mathbb{C})$:*

$$\|A\|_{\mathbb{R}^N \otimes_\varepsilon \mathcal{M}_d^{sa}(\mathbb{C})} \leq \|A\|_G \leq \|A\|_{\mathbb{R}^N \otimes_\pi \mathcal{M}_d^{sa}(\mathbb{C})}$$

avec $(\mathbb{R}^N, \|\cdot\|_G)$ and $(\mathcal{M}_d^{sa}(\mathbb{C}), \|\cdot\|_\infty)$.

Dans ce contexte la norme projective et injective est donné par:

$$\|A\|_{\mathbb{R}^N \otimes_\pi \mathcal{M}_d^{sa}(\mathbb{C})} := \inf \Big\{ \sum_{i=1}^{k} \|p_i\|_G \, \|X_i\|_\infty, \; A = \sum_{i=1}^{k} p_i \otimes X_i \Big\}.$$
$$\|A\|_{\mathbb{R}^N \otimes_\varepsilon \mathcal{M}_d^{sa}(\mathbb{C})} := \sup \Big\{ \langle \pi \otimes \alpha, A \rangle ; \; \|\pi\|_G^* \leq 1, \; \|\alpha\|_1 \leq 1 \Big\}.$$

avec $\mathcal{M}_d^{sa}(\mathbb{C}) \ni \alpha \to \|\alpha\|_1 = \mathrm{Tr}\,|\alpha|$ est la la Schatten 1-norm. Dans le cadre des jeux XOR où les mesures d'Alice sont fixes, si Alice veut savoir si ses mesures sont locales, elle devra calculer la norme $\|\cdot\|_G$ et si cette norme est inférieure ou égale au biais classique $\beta(G)$ alors nous disons que ses mesures sont $G-$Bell-locales. Afin de savoir si ses mesures sont compatibles, elle calculera la norme du tenseur de compatibilité $\|\cdot\|_c$. Le problème de la compréhension du lien entre l'incompatibilité des mesures quantiques et la violation de l'inégalité de Bell devient naturel, dans le sens où Alice doit comparer les deux normes. Nous commençons par une reformulation, en utilisant le langage des normes tensorielles, du fait bien établi suivant : une *violation de l'inégalité de Bell M* observée implique nécessairement la *incompatibilité* des mesures d'Alice. Mathématiquement, cela correspond à la limitation supérieure de la norme de localité de Bell M des mesures d'Alice par leur norme de compatibilité.

Theorem 10.5.9. *Soit un jeu non local inversible décrit par la matrice $G \in \mathcal{M}_N(\mathbb{R})$. Alors pour tout N-uplets de matrice auto-adjointe $A = (A_1, \ldots, A_N)$, on a*

$$\|A\|_G \leq \|A\|_c \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} = \|A\|_c\, \beta(G).$$

Theorem 10.5.10. *Soit un jeu non local inversible décrit par la matrice $G \in \mathcal{M}_N(\mathbb{R})$. Alors, pour tout N-uplets de matrice auto-adjointe $A = (A_1, \ldots, A_N)$, on a*

$$\|A\|_c \leq \|A\|_G \|G^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N}. \tag{10.2}$$

Ces deux théorèmes montrent, que l'égalité suivante $\|\cdot\|_G = \|\cdot\|_c$. n'est pas toujours vraie. Ainsi on retrouve que pour le jeu CHSH donné par:

$$G_{\text{CHSH}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

on a

$$\beta(G_{\text{CHSH}}) = 1 \qquad \text{and} \qquad (G_{\text{CHSH}})^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Ainsi à partir des deux théorèmes on a l'égalité suivante:

$$\|\cdot\|_c = \|\cdot\|_{G_{\text{CHSH}}}$$

L'équivalence forte entre incompatibilité et non localité consiste à l'égalité entre $\|\cdot\|_c$ et $\|\cdot\|_G$. Ainsi on se pose la question, pour quel type de jeu G on peut retrouver cette équivalence stricte. Notez que pour n'importe quel jeu inversible G avec un $N-$uplets de mesure fixe donné par A on a:

$$\|A\|_G \leq \|A\|_c \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \leq \|A\|_G \|G^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N},$$

ainsi on a:

$$\|G^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} \geq 1. \tag{10.3}$$

A fin d'obtenir l'égalité suivante $\|\cdot\|_G = \|\cdot\|_c$, il suffit de normaliser

$$\beta(G) = \|G\|_{\ell_1^N \otimes_\varepsilon \ell_1^N} = 1 \qquad \text{and} \qquad \|G^{-1}\|_{\ell_\infty^N \otimes_\varepsilon \ell_\infty^N} = 1.$$

A une constante prés, ceci est équivalent à une saturation de l'inégalité donné ci-dessus.

Theorem 10.5.11. *L'unique jeu inversible $G \in \mathcal{M}_N(\mathbb{R})$ satisfait*

$$\|G^{-1}\|_{\ell_\infty^2 \otimes_\varepsilon \ell_\infty^2} \|G\|_{\ell_1^2 \otimes_\varepsilon \ell_1^2} = 1$$

sont des jeux à (N=2) questions proportionnel au jeu CHSH: $G = aG_{CHSH}$ avec $a \neq 0$. Donc le jeu CHSH (et ces permutations) donnant une inégalité stricte entre l'incompatibilité de mesures quantiques et la violation des inégalités de Bell.

10.6 Conclusion

En résumé, l'(in)compatibilité et la non-localité sont deux concepts fondamentaux non équivalents. Nous avons introduit la compatibilité des mesures quantiques où nous avons introduit la *dimension de compatibilité* comme nouveau concept pour comprendre l'effet de la dimension de l'espace de Hilbert sur la compatibilité des mesures quantiques. Comme cela a été décrit, afin de rendre compatibles des mesures incompatibles, on peut ajouter du bruit, plusieurs modèles de bruit ayant été établis dans la littérature. Dans un projet en cours, nous introduisons un tel modèle de bruit basé sur un processus de mesure indirect. Au cours de ce processus, l'état quantique est couplé à une sonde, et l'évolution totale est prise comme aléatoire. La mesure sur la sonde induira un POVM bruyant effectif, où le paramètre de bruit est complètement encodé dans la sonde. À partir de cette expérience de pensée, nous trouvons des POVM effectifs bruités différents de ceux introduits dans la littérature ; nous étudions comment ce type de modèle de bruit affecte la compatibilité des POVM. Nous avons développé un nouveau cadre pour unifier la (in)compatibilité de la mesure quantique et la non-localité, basé sur le cadre des jeux non-locaux (jeux XOR) et des normes tensorielles. Les appareils de mesure d'Alice sont fixes, et elle calcule deux normes décrivant respectivement l'(in)compatibilité de ses mesures et la non-localité. En comparant les normes, les seuls jeux satisfaisant l'égalité entre les deux normes sont (au sens fort) le jeu CHSH et ses permutations.

Pour conclure, l'incompatibilité des mesures quantiques et la non-localité quantique sont des sujets fascinants pour comprendre les limites de la théorie quantique. De plus, plusieurs directions peuvent être explorées pour des recherches futures. Dans ce qui suit, nous donnons quelques extensions et directions de recherche qui peuvent être abordées :

- Une question naturelle à poser est la suivante : si l'inversibilité du jeu est une condition nécessaire, peut-on trouver des jeux non inversibles qui donnent l'équivalence entre l'incompatibilité et la non-localité ?
- Dans [9] l'équivalence forte entre l'incompatibilité et la non-localité est donnée par l'égalité de $\|\cdot\|_c = \|\cdot\|_G$; pour la satisfaire, il suffit de saturer l'inégalité donnée par l'équation (10.3). Peut-on relaxer cette condition pour trouver d'autres matrices satisfaisant l'inégalité (10.3)?
- Peut-on étendre les résultats obtenus pour des jeux non locaux avec N questions et M réponses ? Peut-on définir une norme de compatibilité pour de tels jeux M-résultats avec $M \geq 3$?
- Il a été montré dans [91, 92] que la plus grande violation de Bell diverge avec le nombre de questions. On peut se demander si l'un des deux joueurs utilise des mesures incompatibles, comment cela affectera la violation de Bell?
- Les auteurs dans [46, 49] ont montré que si nous considérons des jeux avec trois joueurs, N questions et deux réponses, la violation de Bell maximale diverge avec le nombre de questions. On peut se demander si l'on fixe l'appareil de mesure d'un des joueurs, comment cela affectera quantitativement la quantité par laquelle une inégalité de Bell est violée ?

Resumé en français

L'un des concepts majeurs introduit par la théorie quantique est celui de la compatibilité des mesures quantiques. Il existe certains types de mesures qui ne peuvent pas être effectuées en même temps. Ainsi, on dit que les mesures sont compatibles si on peut mesurer en même temps et d'autre sont incompatibles. L'autre concept majeur de la mécanique quantique est celui de la non-localité qui est l'un des concepts les plus contre-intuitifs de la physique quantique. Ce concept majeur est dû à John Bell qui a montré que la mécanique quantique est intrinsèquement non locale. Ainsi, on parle de violation des inégalités de Bell par la mécanique quantique. Aujourd'hui, la non-localité est comprise à travers les jeux non-locaux. Un jeux non-local consiste en deux joueurs ou plus Alice et Bob et un arbitre. Ce dernier posera un certain nombre de questions aux joueurs qui devront générer un certain nombre de réponses en utilisant une stratégie classique ou quantique. Il se trouve que le maximum des réponses qu'Alice et Bob peuvent générer est intrinsèquement relié à une norme tensorielle caractérisant le jeu. Dans ce formalisme, l'utilisation des stratégies classiques est reliée à la norme de la matrice du jeux lui-même, ainsi la violation des inégalités de Bell se traduit par une inégalité stricte entre les normes tensorielles. Le but de cette thèse consiste à comprendre l'incompatibilité des mesures quantiques ainsi que le lien avec les inégalités de Bell. Dans un premier temps, nous avons introduit la compatibilité des mesures quantiques sous un nouveau point de vue, et analysé les types de bruit qu'on peut effectuer à fin de rendre le système compatible. Ce nouveau point de vue consiste à comprendre et à analyser l'effet de la dimension de l'espace de Hilbert sur l'incompatibilité des mesures. Par ailleurs à fin de rendre des mesures compatibles, on peut introduire l'effet d'un bruit. Comme application, certains états connus sous le nom de MUB sont de nature incompatible, on montre même si on rajoute du bruit aux MUB celle-ci restant incompatible, il existe une isométrie et un espace de Hilbert de dimension plus petite rendant les MUB compatibles. Dans un deuxième temps, nous avons analysé le lien intrinsèque reliant l'incompatibilité des mesures quantique et la violation des inégalités de Bell. Pour cela, on a considéré le cadre des jeux non locaux, où les mesures d'Alice sont fixées. Il est connu qu'une violation des inégalités de Bell nécessite l'utilisation des mesures incompatibles. Par ailleurs, si Alice veut savoir si elle observera une violation des inégalités de Bell si elle utilise des mesures incompatibles. Pour cela, elle doit calculer deux normes tensorielles d'un tenseur construit à partir de ses mesures. Ces normes tensorielles vont caractériser d'une part la compatibilité des mesures d'Alice et d'autre part la violation des inégalités de Bell. Dans ce cadre naturel, comprendre le lien entre incompatibilité des mesures quantiques et de la violation des inégalités de Bell, revient à comparer les deux normes tensorielles. Or, il se trouve que pour le jeu CHSH ces deux normes sont égales, mais on peut montrer généralement qu'elles ne le sont pas. On peut se demander s'il existe d'autre types de jeux satisfaisant cette égalité des normes tensorielles ? Il se trouve que nous avons montré qu'avec des conditions suffisantes, seul le jeu CHSH à constante multiplicative près donne l'égalité entre les normes tensorielles.

English summary

One of the major concepts introduced by quantum theory is the compatibility of quantum measurements. There are certain types of measurements that cannot be made at the same time. Thus,we say that measurements are compatible if they can be measured at the same time and others are incompatible. The other major concept of quantum mechanics is that of nonlocality which is one of the most counterintuitive concepts of quantum physics. This major concept is due to John Bell who showed that quantum mechanics is intrinsically non-local. Thus, we speak of violation of Bell's inequalities by quantum mechanics. Today, nonlocality is understood through nonlocal games. A nonlocal game consists of two or more players Alice and Bob playing against a referee. The referee will ask a number of questions to the players who will have to generate a number of answers using a classical or quantum strategy. It turns out that the maximum number of answers that Alice and Bob can generate is intrinsically linked to a tensor norm characterizing the game. In this formalism, the use of classical strategies is related to the norm of the matrix of the game itself, so the violation of Bell's inequalities results in a strict inequality between the tensor norms. The aim of this thesis is to understand the incompatibility of quantum measures and the link with Bell's inequalities. First, we introduced the compatibility of quantum measures from a new point of view, and analyzed the types of noise that can be made to make the system compatible. This new point of view consists in understanding and analyzing the effect of the dimension of the Hilbert space on the incompatibility of measurements. Moreover, in order to make the measurements compatible, we can introduce the effect of a noise. As an application, some states known as MUB are incompatible in nature, we show that even if we add noise to the MUB it remains incompatible, there is an isometry and a Hilbert space of smaller dimension making the MUB compatible. In a second step, we have analyzed the intrinsic link between the incompatibility of quantum measurements and the violation of Bell's inequalities. For this purpose, we considered the framework of non-local games, where Alice's measurements are fixed. It is known that a violation of Bell's inequalities requires the use of incompatible measurements. On the other hand, if Alice wants to know if she will observe a violation of Bell's inequalities if she uses incompatible measures. To do this, she must compute two tensor norms of a tensor constructed from her measurements. These tensorial norms will characterize on the one hand the compatibility of Alice's measurements and on the other hand the violation of Bell's inequalities. In this natural framework, to understand the link between the incompatibility of quantum measurements and the violation of Bell's inequalities, we have to compare the two tensorial norms. Now, it turns out that for the CHSH game these two norms are equal, but it can be shown generally that they are not. We can ask ourselves if there are other types of games satisfying this equality of the tensorial norms? It turns out that we have shown that with sufficient conditions, only the CHSH game with a multiplicative constant gives the equality between the tensor norms.

Bibliography

1. Loulidi, F. & Nechita, I. The compatibility dimension of quantum measurements. *Journal of Mathematical Physics* **62,** 042205 (2021).
2. Loulidi, F. & Nechita, I. Measurement Incompatibility versus Bell Nonlocality: An Approach via Tensor Norms. *PRX Quantum* **3,** 040325. https://link.aps.org/doi/10.1103/PRXQuantum.3.040325 (4 Dec. 2022).
3. Schrödinger, E. An undulatory theory of the mechanics of atoms and molecules. *Physical review* **28,** 1049 (1926).
4. Heisenberg, W. in *Original Scientific Papers Wissenschaftliche Originalarbeiten* 382–396 (Springer, 1985).
5. Neumann, J. v. Wahrscheinlichkeitstheoretischer aufbau der quantenmechanik. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1927,** 245–272 (1927).
6. Von Neumann, J. *Mathematische grundlagen der quantenmechanik* (Springer-Verlag, 2013).
7. Dirac, P. A. M. The fundamental equations of quantum mechanics. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **109,** 642–653 (1925).
8. Dirac, P. A. M. On the theory of quantum mechanics. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **112,** 661–677 (1926).
9. Wolf, M. M., Pérez-García, D. & Fernández, C. Measurements incompatible in Quantum Theory cannot be measured jointly in any other local theory. *Physical Review Letters* **103,** 230402 (2009).
10. Heinosaari, T. & Ziman, M. *The mathematical language of quantum theory: from uncertainty to entanglement* (Cambridge University Press, 2011).
11. Nielsen, M. & Chuang, I. *Quantum Computation and Quantum Information: 10th Anniversary Edition* ISBN: 9781139495486. https://books.google.fr/books?id=-s4DEy7o-a0C (Cambridge University Press, 2010).
12. Wilde, M. *Quantum Information Theory* ISBN: 9781316813300. https://books.google.fr/books?id=gYcHDgAAQBAJ (Cambridge University Press, 2017).
13. Petz, D. *Quantum information theory and quantum statistics* (Springer Science & Business Media, 2007).
14. Watrous, J. *The theory of quantum information* (Cambridge university press, 2018).
15. Busch, P., Lahti, P., Pellonpää, J.-P. & Ylinen, K. *Quantum measurement* (Springer, 2016).
16. Bratteli, O. & Robinson, D. W. *Operator algebras and quantum statistical mechanics: Volume 1: C*-and W*-Algebras. Symmetry Groups. Decomposition of States* (Springer Science & Business Media, 2012).

17. Takesaki, M. *Theory of Operator Algebras I* ISBN: 9783540422488. https://books.google.fr/books?id=dTnq4hjjtgMC (Springer Berlin Heidelberg, 2001).

18. Haag, R. *Local Quantum Physics: Fields, Particles, Algebras* ISBN: 9783642614583. https://books.google.fr/books?id=OlLmCAAAQBAJ (Springer Berlin Heidelberg, 2012).

19. Wolf, M. M. Quantum channels & operations: Guided tour. *Lecture notes available at http://www-m5. ma. tum. de/foswiki/pub M* **5** (2012).

20. Plávala, M. *General probabilistic theories: An introduction* 2021. https://arxiv.org/abs/2103.07469.

21. Choi, M.-D. Completely positive linear maps on complex matrices. *Linear algebra and its applications* **10,** 285–290 (1975).

22. Aubrun, G. & Szarek, S. J. *Alice and Bob meet Banach* (American Mathematical Soc., 2017).

23. Wootters, W. K. & Zurek, W. H. A single quantum cannot be cloned. *Nature* **299,** 802–803 (1982).

24. Drago, A. *A Review of The Algebraic Approaches to Quantum Mechanics. Appraisals on Their Theoretical Relevance* 2021. https://arxiv.org/abs/2102.00861.

25. Plá vala, M. Conditions for the compatibility of channels in general probabilistic theory and their connection to steering and Bell nonlocality. *Physical Review A* **96.** https://doi.org/10.1103%2Fphysreva.96.052127 (Nov. 2017).

26. David, F. *A short introduction to the quantum formalism[s]* 2012. https://arxiv.org/abs/1211.5627.

27. Witten, E. APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory. *Reviews of Modern Physics* **90.** https://doi.org/10.1103%2Frevmodphys.90.045003 (Oct. 2018).

28. Bluhm, A., Jenčová, A. & Nechita, I. Incompatibility in General Probabilistic Theories, Generalized Spectrahedra, and Tensor Norms. *Communications in Mathematical Physics* **393,** 1125–1198. https://doi.org/10.1007%2Fs00220-022-04379-w (June 2022).

29. Busch, P., Heinosaari, T., Schultz, J. & Stevens, N. Comparing the degrees of incompatibility inherent in probabilistic physical theories. *EPL (Europhysics Letters)* **103,** 10002. https://doi.org/10.1209%2F0295-5075%2F103%2F10002 (July 2013).

30. Aubrun, G., Lami, L., Palazuelos, C. & Plávala, M. Entangleability of cones. *Geometric and Functional Analysis.* https://doi.org/10.1007%2Fs00039-021-00565-5 (May 2021).

31. Albiac, F. & Kalton, N. J. *Topics in Banach space theory* (Springer, 2006).

32. Megginson, R. E. *An introduction to Banach space theory* (Springer Science & Business Media, 2012).

33. Ryan, R. A. & a Ryan, R. *Introduction to tensor products of Banach spaces* (Springer, 2002).

34. Defant, A. & Floret, K. *Tensor norms and operator ideals* (Elsevier, 1992).

35. Rudin, W. & Abouhazim, A. *Analyse fonctionnelle* ISBN: 9782840741138. https://books.google.fr/books?id=Psv-PQAACAAJ (Ediscience international, 1995).

36. Grothendieck, A. *Résumé de la théorie métrique des produits tensoriels topologiques* (Soc. de Matemática de São Paulo, 1956).

37. Grothendieck, A. & Grothendieck, A. *Produits tensoriels topologiques et espaces nucléaires* (American Mathematical Society Providence, 1955).

38. Pisier, G. Grothendieck's theorem, past and present. *Bulletin of the American Mathematical Society* **49,** 237–323 (2012).

39. Friedland, S., Lim, L.-H. & Zhang, J. An elementary and unified proof of Grothendieck's inequality. *L'Enseignement Mathématique* **64,** 327–351 (2019).

40. Einstein, A., Podolsky, B. & Rosen, N. Can quantum-mechanical description of physical reality be considered complete? *Physical review* **47,** 777 (1935).

41. Bell, J. S. On the einstein podolsky rosen paradox. *Physics Physique Fizika* **1,** 195 (1964).

42. Aspect, A. Bell's inequality test: more ideal than ever. *Nature* **398,** 189–190 (1999).

43. Hensen, B. *et al.* Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres. *Nature* **526,** 682–686 (2015).

44. Pironio, S. Lifting Bell inequalities. *Journal of Mathematical Physics* **46,** 062112. https://doi.org/10.1063%2F1.1928727 (June 2005).

45. Brunner, N., Cavalcanti, D., Pironio, S., Scarani, V. & Wehner, S. Bell nonlocality. *Reviews of Modern Physics* **86,** 419–478. https://doi.org/10.1103%2Frevmodphys.86.419 (Apr. 2014).

46. Briët, J. & Vidick, T. Explicit lower and upper bounds on the entangled value of multiplayer XOR games. *Communications in Mathematical Physics* **321,** 181–207 (2013).

47. Landau, L. J. On the violation of Bell's inequality in quantum theory. *Physics Letters A* **120,** 54–56. ISSN: 0375-9601. https://www.sciencedirect.com/science/article/pii/0375960187900752 (1987).

48. Tsirelson, B. S. Some results and problems on quantum Bell-type inequalities. *Hadronic Journal Supplement* **8,** 329–345 (1993).

49. Pérez-Garcıéa, D., Wolf, M. M., Palazuelos, C., Villanueva, I. & Junge, M. Unbounded violation of tripartite Bell inequalities. *Communications in Mathematical Physics* **279,** 455–486 (2008).

50. Pisier, G., Cassels, J., Society, L. M. & Hitchin, N. *Introduction to Operator Space Theory* ISBN: 9780521811651. https://books.google.fr/books?id=0pKL-o7WU0AC (Cambridge University Press, 2003).

51. June 2019. https://doi.org/10.19086%2Fda.8805.

52. Pisier, G. *Tripartite Bell inequality, random matrices and trilinear forms* 2012. https://arxiv.org/abs/1203.2509.

53. Ji, Z., Natarajan, A., Vidick, T., Wright, J. & Yuen, H. *MIP*=RE* 2020. https://arxiv.org/abs/2001.04383.

54. Capraro, V. *A Survey on Connes' Embedding Conjecture* 2010. https://arxiv.org/abs/1003.2076.

55. Kirchberg, E. On non-semisplit extensions, tensor products and exactness of groupC*-algebras. *Inventiones mathematicae* **112,** 449–489 (1993).

56. Fritz, T. Tsirelson's problem and Kirchberg's conjecture. *Reviews in Mathematical Physics* **24,** 1250012 (2012).

57. Junge, M. *et al.* Connes' embedding problem and Tsirelson's problem. *Journal of Mathematical Physics* **52,** 012102 (2011).

58. Vidick, T. MIP= RE.

59. Murray, F. J. & von Neumann, J. On rings of operators. IV. *Annals of Mathematics,* 716–808 (1943).

60. Connes, A. Une classification des facteurs de type III. fr. *Annales scientifiques de l'École Normale Supérieure* **4e série, 6,** 133–252. http://www.numdam.org/articles/10.24033/asens.1247/ (1973).

61. Busch, P. & Lahti, P. J. On various joint measurements of position and momentum observables in quantum theory. *Phys. Rev. D* **29,** 1634–1646. https://link.aps.org/doi/10.1103/PhysRevD.29.1634 (8 Apr. 1984).

62. Busch, P., Schonbek, T. P. & Schroeck Jr, F. E. Quantum observables: compatibility versus commutativity and maximal information. *Journal of mathematical physics* **28,** 2866–2872 (1987).

63. Zhang, Q.-H. & Nechita, I. A Fisher Information-Based Incompatibility Criterion for Quantum Channels. *Entropy* **24,** 805. https://doi.org/10.3390%2Fe24060805 (June 2022).

64. Girard, M., Plávala, M. & Sikora, J. Jordan products of quantum channels and their compatibility. *Nature communications* **12,** 1–6 (2021).

65. Mitra, A. & Farkas, M. Compatibility of quantum instruments. *Physical Review A* **105,** 052202 (2022).

66. Heinosaari, T., Miyadera, T. & Ziman, M. An invitation to quantum incompatibility. *Journal of Physics A: Mathematical and Theoretical* **49,** 123001. https://doi.org/10.1088%2F1751-8113%2F49%2F12%2F123001 (Feb. 2016).

67. Gühne, O., Haapasalo, E., Kraft, T., Pellonpää, J.-P. & Uola, R. Incompatible measurements in quantum information science. *arXiv preprint arXiv:2112.06784* (2021).

68. Heinosaari, T., Miyadera, T. & Ziman, M. An invitation to quantum incompatibility. *Journal of Physics A: Mathematical and Theoretical* **49,** 123001 (2016).

69. Miyadera, T. & Imai, H. Heisenberg's uncertainty principle for simultaneous measurement of positive-operator-valued measures. *Phys. Rev. A* **78,** 052119. https://link.aps.org/doi/10.1103/PhysRevA.78.052119 (5 Nov. 2008).

70. Boyd, S., Boyd, S. P. & Vandenberghe, L. *Convex optimization* (Cambridge university press, 2004).

71. Heinosaari, T., Miyadera, T. & Takakura, R. Testing incompatibility of quantum devices with few states. *Physical Review A* **104.** https://doi.org/10.1103%2Fphysreva.104.032228 (Sept. 2021).

72. Designolle, S., Farkas, M. & Kaniewski, J. Incompatibility robustness of quantum measurements: a unified framework. *New Journal of Physics* **21,** 113053. https://doi.org/10.1088/1367-2630/ab5020 (Nov. 2019).

73. Werner, R. F. Optimal cloning of pure states. *Phys. Rev. A* **58,** 1827–1832. https://link.aps.org/doi/10.1103/PhysRevA.58.1827 (3 Sept. 1998).

74. Keyl, M. & Werner, R. F. Optimal cloning of pure states, testing single clones. *Journal of Mathematical Physics* **40,** 3283–3299. https://doi.org/10.1063%2F1.532887 (July 1999).

75. Studzi ński, M., Ćwikli ński, P., Horodecki, M. & Mozrzymas, M. Group-representation approach to $1 \to N$ universal quantum cloning machines. *Phys. Rev. A* **89,** 052322. https://link.aps.org/doi/10.1103/PhysRevA.89.052322 (5 May 2014).

76. Kay, A. Optimal Universal Quantum Cloning: Asymmetries and Fidelity Measures. https://arxiv.org/abs/1407.4951 (2014).

77. Hashagen, A. K. Universal Asymmetric Quantum Cloning Revisited. https://arxiv.org/abs/1607.03723 (2016).

78. Nechita, I., Pellegrini, C. & Rochette, D. *The Asymmetric Quantum Cloning Region* 2022. https://arxiv.org/abs/2209.11999.

79. Heinosaari, T., Jivulescu, M. A. & Nechita, I. Random positive operator valued measures. *Journal of Mathematical Physics* **61,** 042202 (2020).

80. Bluhm, A. & Nechita, I. Joint measurability of quantum effects and the matrix diamond. *Journal of Mathematical Physics* **59,** 112202 (2018).

81. Bluhm, A. & Nechita, I. Compatibility of quantum measurements and inclusion constants for the matrix jewel. *SIAM Journal on Applied Algebra and Geometry* **4,** 255–296 (2020).

82. Ivanovic, I. Geometrical description of quantal state determination. *Journal of Physics A: Mathematical and General* **14,** 3241 (1981).

83. Durt, T., Englert, B.-G., Bengtsson, I. & Życzkowski, K. On mutually unbiased bases. *International journal of quantum information* **8,** 535–640 (2010).

84. Carmeli, C., Heinosaari, T. & Toigo, A. Informationally complete joint measurements on finite quantum systems. *Physical Review A* **85,** 012109 (2012).

85. Carmeli, C., Heinosaari, T. & Toigo, A. Quantum incompatibility witnesses. *Physical review letters* **122,** 130402 (2019).

86. Bluhm, A., Jenčová, A. & Nechita, I. Incompatibility in general probabilistic theories, generalized spectrahedra, and tensor norms. *Communications in Mathematical Physics* **393,** 1125–1198 (2022).

87. Bluhm, A. & Nechita, I. A tensor norm approach to quantum compatibility. *Journal of Mathematical Physics* **63,** 062201 (2022).

88. Loulidi, F. & Nechita, I. *Measurement incompatibility vs. Bell non-locality: an approach via tensor norms* 2022. https://arxiv.org/abs/2205.12668.

89. Quintino, M. T., Vértesi, T. & Brunner, N. Joint measurability, einstein-podolsky-rosen steering, and bell nonlocality. *Physical review letters* **113,** 160402 (2014).

90. Hirsch, F., Quintino, M. T. & Brunner, N. Quantum measurement incompatibility does not imply Bell nonlocality. *Physical Review A* **97,** 012129 (2018).

91. Junge, M. & Palazuelos, C. Large violation of Bell inequalities with low entanglement. *Communications in Mathematical Physics* **306,** 695–746 (2011).

92. Junge, M., Palazuelos, C., Pérez-Garcıéa, D., Villanueva, I. & Wolf, M. M. Unbounded Violations of Bipartite Bell Inequalities via Operator Space Theory. *Communications in Mathematical Physics* **300,** 715–739. https://doi.org/10.1007%2Fs00220-010-1125-5 (Sept. 2010).

93. Fine, A. Hidden variables, joint probability, and the Bell inequalities. *Physical Review Letters* **48,** 291 (1982).

94. Brunner, N., Cavalcanti, D., Pironio, S., Scarani, V. & Wehner, S. Bell nonlocality. *Reviews of Modern Physics* **86,** 419 (2014).

95. Busch, P., Lahti, P. J. & Mittelstaedt, P. *The quantum theory of measurement* (Springer, 1996).

96. Busch, P., Heinosaari, T., Schultz, J. & Stevens, N. Comparing the degrees of incompatibility inherent in probabilistic physical theories. *EPL (Europhysics Letters)* **103,** 10002 (2013).

97. Designolle, S., Farkas, M. & Kaniewski, J. Incompatibility robustness of quantum measurements: a unified framework. *New Journal of Physics* **21,** 113053 (2019).

98. Heinosaari, T., Schultz, J., Toigo, A. & Ziman, M. Maximally incompatible quantum observables. *Physics Letters A* **378,** 1695–1699 (2014).

99. Carmeli, C., Heinosaari, T. & Toigo, A. State discrimination with postmeasurement information and incompatibility of quantum measurements. *Physical Review A* **98,** 012126 (2018).

100. Heinosaari, T., Reitzner, D. & Stano, P. Notes on joint measurability of quantum observables. *Foundations of Physics* **38,** 1133–1147 (2008).

101. Werner, R. F. Optimal cloning of pure states. *Physical Review A* **58,** 1827 (1998).

102. Keyl, M. & Werner, R. F. Optimal cloning of pure states, testing single clones. *Journal of Mathematical Physics* **40,** 3283–3299 (1999).

103. Studziński, M., Ćwikliński, P., Horodecki, M. & Mozrzymas, M. Group-representation approach to $1 \to N$ universal quantum cloning machines. *Physical Review A* **89,** 052322 (2014).

104. Kay, A. Optimal Universal Quantum Cloning: Asymmetries and Fidelity Measures. *Quantum Information and Computation* **16,** 0991–1028 (2016).

105. Jenčová, A. Incompatible measurements in a class of general probabilistic theories. *Physical Review A* **98,** 012133 (2018).

106. Oszmaniec, M. & Biswas, T. Operational relevance of resource theories of quantum measurements. *Quantum* **3,** 133 (2019).

107. Skrzypczyk, P., Šupić, I. & Cavalcanti, D. All sets of incompatible measurements give an advantage in quantum state discrimination. *Physical review letters* **122,** 130403 (2019).

108. Uola, R., Kraft, T., Shang, J., Yu, X.-D. & Gühne, O. Quantifying quantum resources with conic programming. *Physical review letters* **122,** 130404 (2019).

109. Kuramochi, Y. Minimal sufficient positive-operator valued measure on a separable Hilbert space. *Journal of Mathematical Physics* **56,** 102205 (2015).

110. Heinosaari, T. & Kuramochi, Y. Post-processing minimal joint observables. *Journal of Physics A: Mathematical and Theoretical* **52,** 065301 (2019).

111. McClellan, J. & Parks, T. Eigenvalue and eigenvector decomposition of the discrete Fourier transform. *IEEE Transactions on Audio and Electroacoustics* **20,** 66–74 (1972).

112. Grünbaum, F. A. The eigenvectors of the discrete Fourier transform: A version of the Hermite functions. *Journal of Mathematical Analysis and Applications* **88,** 355–363 (1982).

113. Fendler, G. & Kaiblinger, N. Discrete Fourier transform of prime order: Eigenvectors with small support. *Linear Algebra and its Applications* **438,** 288–302 (2013).

114. Wootters, W. & Fields, B. Optimal state-determination by mutually unbiased measurements. English (US). *Annals of Physics* **191,** 363–381. ISSN: 0003-4916 (May 1989).

115. Klappenecker, A. & Rötteler, M. *Constructions of mutually unbiased bases* in *International Conference on Finite Fields and Applications* (2003), 137–144.

116. Combescure, M. The mutually unbiased bases revisited. *Contemporary Mathematics* **447,** 29 (2007).

117. Knill, E., Laflamme, R. & Viola, L. Theory of quantum error correction for general noise. *Physical Review Letters* **84,** 2525 (2000).

118. Li, C.-K. & Poon, Y.-T. Generalized numerical ranges and quantum error correction. *Journal of Operator Theory,* 335–351 (2011).

119. Choi, M.-D., Kribs, D. W. & Życzkowski, K. Higher-rank numerical ranges and compression problems. *Linear algebra and its applications* **418,** 828–839 (2006).

120. Tverberg, H. A generalization of Radon's theorem. *J. London Math. Soc* **41,** 123–128 (1966).

121. Bárány, I., Blagojevic, P. & Ziegler, G. M. Tverberg's Theorem at 50: Extensions and Counterexamples. *Notices of the AMS* **63** (2016).

122. Weinberg, S. *The quantum theory of fields. Vol. 1: Foundations* (Cambridge University Press, 1995).

123. Kunjwal, R., Heunen, C. & Fritz, T. Quantum realization of arbitrary joint measurability structures. *Physical Review A* **89,** 052126 (2014).

124. Bell, J. S. On the Einstein Podolsky Rosen paradox. *Physics Physique Fizika* **1,** 195–200. https://link.aps.org/doi/10.1103/PhysicsPhysiqueFizika.1.195 (3 Nov. 1964).

125. Aspect, A., Grangier, P. & Roger, G. Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities. *Phys. Rev. Lett.* **49,** 91–94. https://link.aps.org/doi/10.1103/PhysRevLett.49.91 (2 July 1982).

126. Palazuelos, C. & Vidick, T. Survey on nonlocal games and operator space theory. *Journal of Mathematical Physics* **57,** 015220 (2016).

127. Clauser, J. F., Horne, M. A., Shimony, A. & Holt, R. A. Proposed experiment to test local hidden-variable theories. *Physical review letters* **23,** 880 (1969).

128. Bene, E. & Vértesi, T. Measurement incompatibility does not give rise to Bell violation in general. *New journal of Physics* **20,** 013021 (2018).

129. Nielsen, M. A. & Chuang, I. L. *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).

130. Heisenberg, W. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik* **43,** 172–198. ISSN: 0044-3328. https://doi.org/10.1007/BF01397280 (1927).

131. Bohr, N. The quantum postulate and the recent development of atomic theory. *Nature* **121,** 580–590 (1928).

132. Heinosaari, T., Jivulescu, M. A. & Nechita, I. Order preserving maps on quantum measurements. *arXiv preprint arXiv:2202.00725* (2022).

133. Regev, O. & Vidick, T. Quantum XOR games. *ACM Transactions on Computation Theory (ToCT)* **7,** 1–43 (2015).

134. Tsirel'son, B. S. Quantum analogues of the Bell inequalities. The case of two spatially separated domains. *Journal of Soviet Mathematics* **36,** 557–570 (1987).

135. Busch, P. Unsharp reality and joint measurements for spin observables. *Phys. Rev. D* **33,** 2253–2261 (1986).

136. Brougham, T. & Andersson, E. Estimating the expectation values of spin-1/2 observables with finite resources. *Physical Review A* **76,** 052313 (2007).

137. Lawson, T., Linden, N. & Popescu, S. Biased nonlocal quantum games. *arXiv preprint arXiv:1011.6245* (2010).

138. Aubrun, G., Lami, L., Palazuelos, C., Szarek, S. J. & Winter, A. Universal gaps for XOR games from estimates on tensor norm ratios. *Communications in Mathematical Physics* **375,** 679–724 (2020).